The Stability of Cauchy's Equation in Banach Spaces

Charles H. Morgan, Jr.
morgan@math.msu.edu

April 28, 1995

The Stability of the Cauchy Equation in Banach Spaces

Charles H. Morgan, Jr.

A Thesis
Submitted to the Faculty of the
Department of Mathematics of the University of Louisville
in partial fulfillment of the requirements
for the degree of Bachelor of Arts with Honors in Mathematics

Department of Mathematics
University of Louisville
Louisville, Kentucky

April 28, 1995

Author is currently at Michigan State University, East Lansing, Michigan.
E-mail: morgan@math.msu.edu

The Stability of the Cauchy Equation in Banach Spaces

Charles H. Morgan, Jr.

A Thesis Approved on
May 12, 1995
by the Following Reading Committee:

Bruce R. Ebanks
Thesis Director

Thomas Riedel

Prosanna K. Sahoo

CONTENTS

CHAPTER

ONE

Stability

The stability of mathematical equations can be an interesting concept for mathematicians, but it can also be of importance to persons who work outside the realm of pure mathematics. For example, physicists are interested in the stability of the mathematical formulae which they use to model physical processes. More precisely, physicists and other scientists are interested in determining when a small change in an equation used to model a phenomenon gives a large change in the results predicted by the equation and thereby makes the mathematical model unreliable. Mathematicians and the scientists who use their results wish to determine when making small changes in an equation causes its solution to jump far away from the solution of the original equation. We would like to know when small changes in a particular functional equation—Cauchy's fundamental functional equation—have only small effects on its solutions.

This is the essence of the question of stability. It is of tremendous importance to the scientists who often construct their mathematical models experimentally. In such cases, it is extremely difficult if not impossible to derive the precise equation which exactly models a physical process. Often variables with minute effects on the system are difficult to discover under the conditions of the experiment, and so they are left out of the mathematical model. Is it possible, though, that omitting such a variable could make the solutions to the model completely unreliable in describing the very process whose events it was designed to predict, if conditions of the experiment are changed even only slightly?

The question could be and has been raised about how to define stability for an equation or system of equations. Which definition of stability is appropriate for a given problem? Throughout this work, we will see and use a number of different definitions of stability. In order to establish a clear concept of what stability is, we examine the idea as it is applied to the solutions of differential equations and to functional iteration.

In the field of differential equations, a type of stability named for the mathematician Liapunov is often considered. Here one includes with a differential equation its set of initial conditions, say

$x(t_0) = x_0$. Suppose that an exact solution $x(t; t_0, x_0)$ of the differential equation exists on the open infinite interval (t_0, ∞) and that the solution x is continuous on (t_0, ∞). The solution x is stable in the Liapunov sense if, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x(t; t_0, x_1) - x(t; t_0, x_0)\| < \epsilon$ for all t in the interval (t_0, ∞) whenever x_1 is within δ of the vector x_0 [6]. If we begin with a different initial condition, we get a different solution $x(t; t_0, x_1)$ of the differential equation. Now, if our second intial condition is "close" to the first initial condition, and if the two solutions $x(t; t_0, x_1)$ and $x(t; t_0, x_0)$ of the differential equation remain "close" to each other, then the solution $x(t; t_0, x_0)$ exhibits Liapunov stability.

A similar concept of stability is of importance in the study of dynamical systems (cf. [14]). Often biological and other processes are modeled by the iteration of a particular function; i.e., the repeated composition of the function with itself. In such models, the values of the variables in one state depend entirely upon the values of the variables which occurred in the preceding state; e.g., the population of the earth in the year 2000 depends upon the population of the earth in the year 1999. As the Ukrainian mathematician Sharkovsky showed (cf. [14]), the process of functional iteration can be very complicated since the existence of a three-cycle mandates the existence of cycles of all other orders. In such an instance, one can pick two seed values which are as close as one pleases but eventually end up with values quite far apart after a number of iterations. Will small changes in the iteration function result in small changes in the orbits obtained by the iterative process? A result given by Block (cf. [14]) answers this question somewhat positively. According to Block, if we are given a compact set I and continuous function $f : I \to I$ which has a cycle of order m, there exists a real number $\epsilon > 0$ such that every continuous function $g : I \to I$ which satisfies the inequality $\|f(x) - g(x)\| < \epsilon$ for every $x \in I$ has a cycle of order k following m in the Sharkovsky ordering of cycles. That is to say, that if we begin with a function f which has a fixed number of cycles, we can find within some distance of f another function g which has fewer cycles than does f. Kloeden, on the other hand, proved a result (cf. [14]) which gives a quite negative answer to the question regarding iteration orbits. On a compact set I, for any continuous function $f : I \to I$ and for any $\epsilon > 0$, there exists a chaotic function $g : I \to I$ within ϵ of f for all $x \in I$. Even if the iteration orbits obtained by f are non-chaotic, we can find within any arbitrary distance of f a function g whose orbits are chaotic.

While we will find much more positive results in dealing with our question of the stability of Cauchy's fundamental functional equation, we can easily infer from the above situation why asking questions about stability is of some importance. We get even more justification for examining such questions when we understand how a functional equation which appears quite simple in its construction can have solutions which are far from simple—particularly the solution given by G. Hamel, which we examine in the next section.

In his book *A Collection of Mathematical Problems*, S. M. Ulam posed the question of the stability of the Cauchy equation. Ulam asked: "... if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near to the solutions of the strict equation?" [16] Originally, he had proposed the following more specific question during a lecture given before the University of Wisconsin's Mathematics Club. If a function f approximately satisfies Cauchy's functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

when does there exist an exact solution of (1.1) which f approximates? [10] The problem has been considered for many different types of spaces by a number of writers including D. H. Hyers, Th. M. Rassias, and Z. Gajda. Before we begin looking at the many theorems developed to answer the question of the stability of Cauchy's fundamental functional equation, we should first look at its exact solutions.

CHAPTER

TWO

The Solutions of Cauchy's Equation

In 1821 A. L. Cauchy solved the equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

already known to A. M. Legendre and C. F. Gauss as much as thirty years earlier. This functional equation has many practical applications (cf. [1], [2], and [4]). In the discussion which follows, we use the following notation: $\mathbb{N}$ denotes the natural numbers, $\mathbb{Q}$ the rational numbers, and $\mathbb{R}$ the real numbers.

Assuming that $f : \mathbb{R} \to \mathbb{R}$ we can proceed toward a solution of (1.1) in the following manner: First, we put $y = x$ to get $f(2x) = 2f(x)$. By induction it is proved that

$$f(nx) = nf(x) \tag{2.1}$$

for any positive integer n. This is accomplished by replacing y by successively higher integer multiples of x in (1.1). Now, if $x = \frac{m}{n}t$, where m and n are positive integers and t is an arbitrary real number, then $nx = mt$, and we can write

$$f(nx) = f(mt). \tag{2.2}$$

Since $m, n \in \mathbb{N}$, then (2.2) can be written as

$$nf(x) = mf(t) \tag{2.3}$$

or

$$f(x) = \frac{m}{n}f(t), \tag{2.4}$$

but since $x = \frac{m}{n}t$, then (2.4) becomes

$$f\left(\frac{m}{n}t\right) = \frac{m}{n}f(t). \tag{2.5}$$

Replacing x and y both by zero in (1.1) gives that $f(0) = 0$. By replacing y by $-x$ in (1.1), we obtain the following equation

$$f(0) = f(x) + f(-x) \tag{2.6}$$

which shows us that f is an odd function since $f(0) = 0$. These last two statements combined show that any solution of (1.1) is $\mathbb{Q}$-homogeneous; i.e., it satisfies

$$f(rt) = rf(t) \tag{2.7}$$

for any rational number r and any real number t.

So far, we have made no assumptions about the solutions of (1.1). We have obtained (2.7) by following only the requirements implicit in the statement of (1.1). In order to arrive at further information about certain types of solutions of (1.1), we must make some assumptions about their nature; i.e., we must assume some type of regularity condition about f.

G. Darboux showed that if f is continuous at a single point in its domain, then it must be continuous everywhere. Later, Darboux was able to show that weaker regularity assumptions on f could also force its continuity on its entire domain. For example, if it is assumed that f is nonnegative (or nonpositive) for sufficiently small positive x, then f is continuous everywhere. Also, if f is bounded on an arbitrarily small interval then f is continuous. In 1929, A. Ostrowski proved that the upper- or lower-boundedness of f on a set of positive measure was sufficient to guarantee its continuity.

Nothing in the statement of (1.1), however, requires the continuity of its solutions. Only the linearity of the solutions on the rationals is inherent for (1.1). Do discontinuous solutions of (1.1) exist, and if so, how can one demonstrate their existence? The work already done by Cauchy, Darboux, and Ostrowski seemed to indicate that the only solutions of (1.1) are all continuous and linear. In 1905, however, the German mathematician G. Hamel demonstrated the existence of discontinuous solutions of (1.1); however, in order to do so, he relied upon Zermelo's Axiom of Choice (cf. [14]).

Hamel first constructed a basis for the real numbers over the rational numbers. According to Hamel, every nonzero real number can be written uniquely as a finite sum of nonzero rational multiples of elements of the basis. Naturally because there are uncountably many irrational numbers and only countably many rationals, Hamel's basis for the real numbers must have uncountably many elements. Now, if (with some abuse of notation) a basis for the real numbers is $\{h_1, h_2, h_3, \ldots\}$, then we can express each nonzero real number x as a finite sum

$$x = r_1h_1 + r_2h_2 + \cdots + r_nh_n \tag{2.8}$$

where each r_i is a rational number and each h_i is distinct from every other.

If H is a Hamel basis for the real numbers over the rationals and f is any mapping of H into the real numbers, then the extension of f defined, for x expressed in (2.8), by

$$f(x) = r_1f(h_1) + r_2f(h_2) + \cdots + r_nf(h_n) \tag{2.9}$$

is a solution of Cauchy's equation (1.1) on $\mathbb{R}$. This is easily enough demonstrated to be true. For any pair of real numbers x and y, the three numbers x, y, and $x+y$ can be expressed as the sum of a finite number of rational multiples of Hamel basis elements. Taking the union of the set of basis elements used to express x, the set of basis elements used to express y, and the set of basis elements used to express $x+y$, we can write x, y, and $x+y$ with rational multiples of the elements of this finite union of basis elements, say $\{h_1, \ldots, h_m\}$, where some of these rational scalars are allowed to be zero. This implies the following equation:

$$\begin{aligned} f(x)+f(y) &= r_1 f(h_1)+\cdots+r_m f(h_m)+s_1 f(h_1)+\cdots+s_m f(h_m) \\ &= (r_1+s_1)f(h_1)+\cdots+(r_m+s_m)f(h_m). \end{aligned} \tag{2.10}$$

But the representation of a real number as in (2.8) is unique up to nonzero rationals. Thus the expression for $x+y$ is

$$(r_1h_1+\cdots+r_mh_m)+(s_1h_1+\cdots+s_mh_m)=(r_1+s_1)h_1+\cdots+(r_m+s_m)h_m,$$

so that (2.10) yields $f(x+y)=f(x)+f(y)$. Conversely, if f is any solution of (1.1), then it can be written in the form given in (2.9). The extension of f given by (2.9) will yield discontinuous solutions of Cauchy's equation if f is any arbitrary mapping of the elements of the Hamel basis into the real numbers such that $f(h_2) \neq \frac{h_2}{h_1} f(h_1)$.

Now that one can demonstrate the existence of discontinuous solutions of Cauchy's equation (1.1), how might one describe their general appearance? Naturally, it is easy to visualize the continuous solutions of Cauchy's equation, but it may not be so easy to describe the solutions envisioned by Hamel. As J. Aczél has shown in [2], if $f:\mathbb{R}\to\mathbb{R}$ is a solution of Cauchy's equation and if f is discontinuous, then its graph is dense in the plane $\mathbb{R}^2$. The proof of this statement involves only linear algebra and is given in the following theorem.

Theorem 2.1 *The graph of every solution $f:\mathbb{R}\to\mathbb{R}$ of*

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

for all $x, y \in \mathbb{R}$, which is not of the form $f(x)=ax$ for all $x\in\mathbb{R}$ and some real number a, is everywhere dense in the real plane $\mathbb{R}^2$.

PROOF. The graph of f is the set of ordered pairs

$$G=\{(x,y)|x\in\mathbb{R}, y=f(x)\in\mathbb{R}\}.$$

Now take some $x_1 \in \mathbb{R}\setminus\{0\}$. If f is not of the form $f(x)=ax$ for any real number a, then there exists a nonzero real number $x_2\in\mathbb{R}$ such that

$$\frac{f(x_1)}{x_1} \neq \frac{f(x_2)}{x_2}. \tag{2.11}$$

The above inequality (2.11) implies

$$x_2 f(x_1) \neq x_1 f(x_2), \tag{2.12}$$

or

$$x_1 f(x_2) - x_2 f(x_1) \neq 0. \tag{2.13}$$

Notice now that (2.13) gives us the following:

$$x_1 f(x_2) - x_2 f(x_1) = \begin{vmatrix} x_1 & f(x_1) \\ x_2 & f(x_2) \end{vmatrix} \neq 0 \tag{2.14}$$

so that the vectors $\mathbf{v_1} = (x_1, f(x_1))$ and $\mathbf{v_2} = (x_2, f(x_2))$ are linearly independent; therefore, they span $\mathbb{R}^2$. We can then write any vector $\mathbf{v} \in \mathbb{R}^2$ as follows:

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} \tag{2.15}$$

for some real numbers c_1 and c_2. There also exists a vector $\mathbf{v'} = r_1\mathbf{v_1} + r_2\mathbf{v_2}$ which is arbitrarily close to vector $\mathbf{v}$ for some rational numbers r_1 and r_2, since we can find a rational number arbitrarily close to any real number; i.e., since the rationals are dense in the reals. Now,

$$\begin{aligned} r_1\mathbf{v_1} + r_2\mathbf{v_2} &= r_1(x_1, f(x_1)) + r_2(x_2, f(x_2)) \\ &= (r_1x_1 + r_2x_2, r_1 f(x_1) + r_2 f(x_2)) \\ &= (r_1x_1 + r_2x_2, f(r_1x_1 + r_2x_2)) \end{aligned} \tag{2.16}$$

where the last equality follows from (2.7) because r_1 and r_2 are rational numbers. Then we have

$$G' = \{(x, y) | x = r_1x_1 + r_2x_2, y = f(x) \text{ and } r_1, r_2 \in \mathbb{Q}\} \tag{2.17}$$

is everywhere dense in $\mathbb{R}^2$. Therefore, G which contains G' is also dense in $\mathbb{R}^2$. ■

The above theorem also demonstrates a reasonable conclusion about the measurable solutions of (1.1). If a solution $f : \mathbb{R} \to \mathbb{R}$ of (1.1) is measurable, then it is of the form $f(x) = cx$, for some constant c. We will find this fact useful later when we examine a few definitions of stability.

CHAPTER

THREE

Hyers' Solution

D. H. Hyers of the University of Wisconsin was the first to provide an answer to Ulam's original question (cf. [10]). Hyers phrased the problem precisely and provided an elegant solution. He began by looking at a particular functional inequality (shown below) in a quite intuitive manner—one which is consistent with the rigorous definitions of continuity and limits of a function. It must be noted by anyone who has attempted a solution of Ulam's problem that it was Hyers who conceived the basic idea used by subsequent writers including this writer. By working to construct a Cauchy sequence of function values, Hyers was able to define directly a particular solution T of Cauchy's equation which an approximately additive function f would approximate. He then proved the uniqueness of the additive function T and showed that it would be continuous whenever f is continuous at some point.

Hyers constructed his solution by relying upon the inherent properties of the inequality

$$\|f(x+y) - f(x) - f(y)\| < \delta. \tag{3.1}$$

Replacing y by x in this inequality gives $\|f(2x) - 2f(x)\| < \delta$. Now dividing this last inequality through by 2 and replacing x by $\frac{x}{2}$ yields $\|\frac{1}{2}f(x) - f(\frac{x}{2})\| < \frac{\delta}{2}$. We now replace x by $\frac{x}{2}$ in this latter inequality to obtain $\|\frac{1}{2}f(\frac{x}{2}) - f(\frac{x}{4})\| < \frac{\delta}{2}$. We continue this process of replacing x by $\frac{x}{2}$ in each of the inequalities obtained until we reach the inequality $\|\frac{1}{2}f(\frac{x}{2^{n-1}}) - f(\frac{x}{2^n})\| < \frac{\delta}{2}$. The idea now is to add the appropriate multiples of all these inequalities together by the triangle inequality so that only the terms involving the nth power of 2 remain. To do this, we also rely upon the following identity which is readily verified by the reader:

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = 2^n - 1. \tag{3.2}$$

Now, we multiply our first inequality by $\frac{1}{2^{n-1}}$, our second inequality by $\frac{1}{2^{n-2}}$, and so forth. Then, when we add the inequalities obtained after this multiplication, we get

$$\frac{1}{2^{n-1}}\left\|\frac{1}{2}f(x)-f\left(\frac{x}{2}\right)\right\|+\cdots+\left\|\frac{1}{2}f\left(\frac{x}{2^{n-1}}\right)-f\left(\frac{x}{2^n}\right)\right\|<\delta\left(\frac{1}{2^n}+\cdots+\frac{1}{2}\right). \tag{3.3}$$

By (3.2) the right side of (3.3) equals $\delta(1-2^{-n})$—a fact that Hyers used to formulate the induction hypothesis used in his theorem which we present now.

Theorem 3.1 *Let E and E' be two Banach spaces, and let $f: E \to E'$ be a mapping such that for some $\delta > 0$*

$$\|f(x+y)-f(x)-f(y)\|<\delta \tag{3.4}$$

for all x and y in E. Then the limit $T(x)=\lim_{n\to\infty}2^{-n}f(2^n x)$ exists for each $x \in E$, the transformation T is additive, and the inequality $\|f(x)-T(x)\|<\delta$ is true for all $x \in E$. Moreover, T is the unique such additive mapping.

Proof. By letting $y = x$ in (3.4), one obtains the inequality $\|f(2x)-2f(x)\|<\delta$. Dividing this last inequality through by 2 and replacing x by $\frac{x}{2}$ yields

$$\|2^{-1}f(x)-f(2^{-1}x)\|<\frac{\delta}{2}. \tag{3.5}$$

Then we make the inductive assumption

$$\|2^{-n}f(x)-f(2^{-n}x)\|<\delta(1-2^{-n}) \tag{3.6}$$

for some positive integer n. Clearly the inductive assumption is true for the case $n = 1$, since replacing n by 1 in (3.6) would give (3.5). Now, the inductive assumption must be demonstrated to hold true for the next positive integer $n+1$. Replacing x by $2^{-n}x$ in (3.5) gives the following result:

$$\|2^{-1}f(2^{-n}x)-f(2^{-n-1}x)\|<\frac{\delta}{2}. \tag{3.7}$$

Multiplying (3.6) by 2^{-1} yields

$$\|2^{-1}f(2^{-n}x)-2^{-n-1}f(x)\|<\delta(2^{-1}-2^{-n-1}). \tag{3.8}$$

Then, adding (3.7) and (3.8) by the triangle inequality gives

$$\|f(2^{-n-1}x)-2^{-n-1}f(x)\|<\delta(1-2^{-n-1}) \tag{3.9}$$

so that the inductive assumption (3.6) is true for any positive integer n.

Now, if $m > n > 0$ then $m - n \in \mathbb{N}$; therefore, n can be replaced by $m - n$ in (3.6), and the following inequality is obtained:

$$\|f(2^{n-m}x)-2^{n-m}f(x)\|<\delta(1-2^{n-m}). \tag{3.10}$$

Dividing (3.10) by 2^n and replacing x by $2^m x$ then yields

$$\|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\| < \delta(2^{-n} - 2^{-m}). \tag{3.11}$$

As $m, n \to \infty$, the right-hand side of (3.11) converges to zero. This then implies that the sequence $\{2^{-n} f(2^n x)\}_{n=1}^{\infty}$ is a Cauchy sequence for every fixed x in E. Since E' is a Banach space, the limit of this sequence exists, and this limit is in E'. Define $T : E \to E'$ by

$$T(x) \equiv \lim_{n\to\infty} 2^{-n} f(2^n x).$$

Dividing (3.4) by 2^{-n} and replacing x and y by $2^n x$ and $2^n y$, respectively, yields

$$\|2^{-n} f\left(2^n(x+y)\right) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y)\| < 2^{-n}\delta. \tag{3.12}$$

As $n \to \infty$, then (3.12) becomes

$$\|T(x+y) - T(x) - T(y)\| = 0 \tag{3.13}$$

or

$$T(x+y) = T(x) + T(y) \tag{3.14}$$

so that the mapping $T : E \to E'$ defined above is itself an additive function; i.e., it is an exact solution of Cauchy's equation (1.1).

Now, replace x by $2^n x$ in (3.6) to obtain

$$\|f(x) - 2^{-n} f(2^n x)\| < \delta(1 - 2^{-n}). \tag{3.15}$$

As $n \to \infty$, then inequality (3.15) becomes

$$\|f(x) - T(x)\| \le \delta. \tag{3.16}$$

Suppose that there exists another mapping $t : E \to E'$ such that

$$\|f(x) - t(x)\| \le \delta \tag{3.17}$$

and such that $t(y) \neq T(y)$ for some $y \in E$. Then there exists some positive integer n such that

$$n > \frac{2\delta}{\|T(y) - t(y)\|}. \tag{3.18}$$

This last inequality implies that $\|T(ny) - t(ny)\| > 2\delta$. However, adding inequalities (3.16) and (3.17) by the triangle inequality and replacing x by ny yields

$$\|t(ny) - T(ny)\| \le 2\delta, \tag{3.19}$$

so that the assumption that t and T were not equal at some point in their common domain contradicts (3.19). It must be true then that the additive mapping T which f approximates is unique. ■

So far Hyers has considered the possibility that T could be either of the two primary types of solutions of Cauchy's equation: either continuous everywhere or discontinuous everywhere. By *reductio ad absurdum* Hyers proves the following theorem regarding the continuity of the additive mapping T:

Theorem 3.2 *Under the hypotheses of Theorem 3.1 suppose that f is continuous at a single point y of E, then T is continuous everywhere on E.*

PROOF. Assume that the additive function T is not continuous at some point y in E. By a result of Darboux (cf. [14]), it must be that T is discontinuous everywhere on E. Then there exists an integer k and a sequence x_n of points in E such that $x_n \to 0$ as $n \to \infty$ and such that $\|T(x_n)\| > \frac{1}{k}$ for all positive integers n. Also, let $m \in \mathbb{N}$ be such that $m > 3k\delta$. Then $\|T(mx_n + y) - T(y)\| = \|T(mx_n)\| > 3\delta$. However, there exists some integer $N \in \mathbb{N}$ such that for $n > N$, $\|T(mx_n+y)-T(y)\| \leq \|T(mx_n+y)-f(mx_n+y)\|+\|f(mx_n+y)-f(y)\|+\|f(y)-T(y)\| < 3\delta$, so that a contradiction is obtained, and the result is established. □

CHAPTER

FOUR

Rassias' Solution

While working as a postdoctoral Research Fellow at the University of California at Berkley in 1977, Th. M. Rassias took up the question of the stability of the Cauchy equation (cf. [12]). He took an approach very similar to that of Hyers, but he chose an inequality which subsumed that of his predecessor. of Hyers. There was, however, one large difference between the requirements of Hyers and Rassias—the latter did not require the boundedness of the Cauchy difference as did the former. Rassias assumed that the norm of the Cauchy difference was majorizable by a function of x and y. Again, the mapping f was assumed to be from one Banach space into another. As in Hyers' proof, Rassias directly defined the additive mapping T to be the limit $T(x) \equiv \lim_{n\to\infty} 2^{-n} f(2^n x)$. The results are given in the following theorem. The inductive assumption which Rassias used does not differ greatly in its construction from that used by Hyers, and so we do not set it up here.

Theorem 4.1 *Let E_1 and E_2 be two Banach spaces, and let $f : E_1 \to E_2$. Assume that there exists $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leq \theta \tag{4.1}$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\frac{\|f(x) - T(x)\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p} \tag{4.2}$$

for all $x \in E_1$.

Proof. Setting $y = x$ in (4.1) yields

$$\frac{\|2^{-1}f(2x) - f(x)\|}{\|x\|^p} \leq \theta. \tag{4.3}$$

Make the inductive assumption

$$\frac{\|2^{-n}f(2^n x)-f(x)\|}{\|x\|^p} \leq \theta \sum_{m=0}^{n-1} 2^{m(p-1)} \tag{4.4}$$

for some positive integer n, and for some $\theta \geq 0$. Clearly, this is true for $n=1$ since setting n equal to 1 in (4.4) gives (4.3). The inductive step must now be demonstrated to hold true for the integer $n+1$. Replacing x by $2x$ in (4.4) gives

$$\frac{\|2^{-n-1}f(2^{n+1}x)-2^{-1}f(2x)\|}{\|x\|^p} \leq \theta \sum_{m=1}^{n} 2^{m(p-1)}. \tag{4.5}$$

Now adding (4.3) and (4.5) by the triangle inequality yields the following result

$$\frac{\|2^{-n-1}f(2^{n+1}x)-f(x)\|}{\|x\|^p} \leq \theta \sum_{m=0}^{n} 2^{m(p-1)} \tag{4.6}$$

so that (4.4) is valid for any positive integer n. Now, the right side of (4.4) is a geometric series and converges for $p<1$. In particular, $\sum_{m=0}^{n-1} 2^{m(p-1)} < \sum_{m=0}^{\infty} 2^{m(p-1)}$ so that (4.4) can be written as

$$\frac{\|2^{-n}f(2^n x)-f(x)\|}{\|x\|^p} \leq \frac{2\theta}{2-2^p} \tag{4.7}$$

for $p<1$. Now, if m is a positive integer such that $m>n>0$, then $m-n$ is also a positive integer and can be substituted for n in (4.7). By doing this, we can rewrite (4.7) as follows:

$$\|2^{n-m}f(2^{m-n}x)-f(x)\| \leq \frac{2\theta\|x\|^p}{2-2^p}. \tag{4.8}$$

Now replacing x by $2^n x$ in (4.8) gives the inequality

$$\|2^{n-m}f(2^m x)-f(2^n x)\| \leq \frac{2^{np}2\theta\|x\|^p}{2-2^p} \tag{4.9}$$

which, when we divide through by 2^n gives us the result

$$\|2^{-m}f(2^m x)-2^{-n}f(2^n x)\| \leq 2^{n(p-1)}\frac{2\theta\|x\|^p}{2-2^p}. \tag{4.10}$$

Taking the limit of (4.10) as m and n both tend toward infinity shows that the right-hand side of (4.10) converges to zero for any $p<1$. This implies then that the sequence $\{2^{-n}f(2^n x)\}_{n=1}^{\infty}$ is a Cauchy sequence. Since E_2 is a Banach space, this sequence converges for each fixed x in E_1 to a limit point in E_2. We call this limit $T(x)$. Now, replacing x and y by $2^n x$ and $2^n y$ in (4.1) and dividing the inequality through by 2^n allows us to write the following:

$$\|2^{-n}f\left(2^n(x+y)\right)-2^{-n}f(2^n x)-2^{-n}f(2^n y)\| \leq 2^{n(p-1)}\theta(\|x\|^p+\|y\|^p). \tag{4.11}$$

Taking the limit of both sides of (4.11) as n tends toward infinity then gives

$$\|T(x+y) - T(x) - T(y)\| = 0 \tag{4.12}$$

or

$$T(x+y) = T(x) + T(y) \tag{4.13}$$

for any x and y in E_1, so that the mapping T defined by $T(x) \equiv \lim_{n\to\infty} 2^{-n} f(2^n x)$ is indeed an additive mapping.

Moreover, if we take the limit of both sides of (4.7) as n tends toward infinity, we get the following

$$\|T(x) - f(x)\| \leq \frac{2\theta\|x\|^p}{2-2^p}, \tag{4.14}$$

which is equivalent to (4.2).

Suppose now that there exists another additive mapping $t : E_1 \to E_2$ such that

$$\|t(x) - f(x)\| \leq \frac{2\theta\|x\|^q}{2-2^q} \tag{4.15}$$

for some q such that $0 \leq q < 1$; however, adding (4.14) and (4.15) by the triangle inequality, we find that

$$\|T(x) - t(x)\| \leq \frac{2\theta\|x\|^p}{2-2^p} + \frac{2\theta\|x\|^q}{2-2^q}. \tag{4.16}$$

Now, replacing x by nx in (4.16) and dividing the inequality through by n gives

$$\|T(x) - t(x)\| = \left\| \frac{1}{n}T(nx) - \frac{1}{n}t(nx) \right\| \leq n^{p-1}\frac{2\theta\|x\|^p}{2-2^p} + n^{q-1}\frac{2\theta\|x\|^q}{2-2^q}. \tag{4.17}$$

Since both $p, q < 1$, taking the limit of both sides of (4.17) as n tends toward infinity gives

$$\lim_{n\to\infty} \|T(x) - t(x)\| = 0 \tag{4.18}$$

for all $x \in E_1$; therefore, $T(x) = t(x)$ for all $x \in E_1$. This establishes the existence and uniqueness of the additive mapping T which f approximates whenever f satisfies (4.1). □

In [12], Rassias considered the continuity of the additive mapping T and the above theorem together; however, we treat them separately in order to preserve the intuitiveness of Rassias' extension of Hyers' Theorems 3.1 and 3.2. In the following theorem, we now consider a condition on the mapping f which is sufficient to guarantee the linearity of the additive mapping T, provided that E_1 and E_2 are linear spaces over the reals.

Theorem 4.2 *Let f be defined as above in Theorem 4.1, but let f satisfy the additional condition that $f(tx)$ is continuous in t for each fixed $x \in E_1$. Assume also that E_1 and E_2 are real Banach spaces. Then T is linear.*

PROOF. As we discussed earlier in this paper, if T is a solution of Cauchy's equation (1.1) for all x and y in E_1, then $T(rx) = rT(x)$ for any rational number r. Choose $x_0 \in E_1$ and $\rho \in E_2^*$ (the dual space of E_2). Let t be a real number, and consider the mapping

$$t \mapsto \rho(T(tx_0)) = \phi(t). \tag{4.19}$$

Then $\phi : \mathbb{R} \to \mathbb{R}$ is also a solution of Cauchy's equation (1.1). Because of the definition for T in the above theorem and since ρ is continuous, we can write ϕ as follows:

$$\phi(t) = \lim_{n\to\infty} 2^{-n}\rho(f(2^n tx_0)), \tag{4.20}$$

and each $\phi_n(t) = 2^{-n}\rho(f(2^n tx_0))$. We note that since we assume that f is continuous in t and since ρ is continuous, each of the functions $\phi_n(t)$ is continous in t. Thus, $\phi(t)$ is the pointwise limit of a sequence of continuous functions; therefore, it is a measurable function. By our discussion of the solutions of Cauchy's equation earlier in this paper, we know that if ϕ is a solution of (1.1) and if ϕ is measurable, then it must be continuous; therefore, $\phi(at) = a\phi(t)$ for any real number a. Since this is true for all $x_0 \in E_1$ and all $\rho \in E_2^*$, we conclude that $T(ax) = aT(x)$ for any $a \in \mathbb{R}$ also, and therefore T is linear. □

Here, we should make a note that the linearity of the additive mapping T does not imply its continuity . Because the spaces E_1 and E_2 could be infinite-dimensional, the mapping T could be linear but not continuous. We take the following example. Let X be the space of polynomials on the interval $[0, 1]$ over the reals. This is an infinite-dimensional space. We define the norm as follows:

$$\|x(t)\| = max_{t\in[0,1]}|x(t)|$$

for each polynomial $x(t)$. Also, let $T : X \to X$ be the mapping defined by

$$T\,(x(t)) = x'(t).$$

Because the derivative is a linear operator, T is a linear mapping. We shall now demonstrate by *reductio ad absurdum* that it is not a continuous mapping.

We assume that T is a continuous mapping. Hence given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|T(x) - T(x_0)\| < \epsilon \text{ whenever } \|x - x_0\| < \delta.$$

Let $x_0 = 0$, so that the above definition of continuity reduces to

$$\|T(x)\| < \epsilon \text{ whenever } \|x\| < \delta.$$

If we let the polynomial $x(t) = \lambda t^n$, then $T(x) = T(\lambda t^n) = \lambda n t^{n-1}$. Choosing $\lambda = \frac{\delta}{2}$, we obtain from the above inequality

$$\|\lambda n t^{n-1}\| = |\lambda| n = \frac{\delta n}{2} < \epsilon \text{ whenever } \|\lambda t^n\| = |\lambda| = \frac{\delta}{2} < \delta.$$

This implies that $\frac{\delta n}{2} < \epsilon$ for all n, which is false. Hence, our assumption that the linear mapping T is also continuous is not true.

CHAPTER

FIVE

Gajda's Solution

Some years after the publication [12] of his solution of Ulam's question of stability of Cauchy's fundamental functional equation, Rassias noticed that it was unnecessary to assume $p \in [0, 1)$, but that $p < 1$ was sufficient to guarantee the results of Theorem 4.1. Therefore, Theorem 4.1 above can be written in a more general manner as follows:

Theorem 5.1 *Let E_1 and E_2 be two Banach spaces, and let $f : E_1 \to E_2$. Assume that there exists $\theta \geq 0$ and $p < 1$ such that*

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leq \theta \tag{5.1}$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\frac{\|f(x) - T(x)\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p} \tag{5.2}$$

for all $x \in E_1$.

During the Twenty-Seventh International Symposium on Functional Equations, Rassias asked whether the theorem could be extended to include the case for $p \geq 1$ [11]. Z. Gajda of Katowice provided an answer to the question along with a counterexample for the case of $p = 1$ [8]—a counterexample which we shall examine later in this paper. In order to extend the theorem to include $p > 1$, Gajda gave a new definition for the additive mapping T. While Hyers and Rassias had defined $T(x)$ to be the limit of the sequence $\{2^{-n} f(2^n x)\}_{n=1}^{\infty}$ when $p < 1$, Gajda defined $T(x)$ to be the limit of the sequence $\{2^n f(2^{-n} x)\}_{n=1}^{\infty}$ for each fixed x and proved that it was indeed a Cauchy sequence, when $p > 1$.

Gajda also noted that the sequential completeness of the domain space E_1 was an assumption not needed in the proof of Rassias' theorem; therefore, it could be removed. J. Rätz [13] showed also that the conditions on the domain space could be relaxed even further. He required only that E_1 be a vector space whose scalar field includes the rational numbers. Notice that throughout the proofs of Hyers' and Rassias' theorems, we used only those scalar multiples which were powers of 2, and the smallest field from which such numbers can come is the field of all rationals.

Using Rassias' inequality (4.1), we shall make some replacements for x and y in order to set up our induction hypothesis. Replacing x and y by $\frac{x}{2}$ in (4.1) we get

$$\left\| f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq 2^{1-p}\theta\|x\|^p. \tag{5.3}$$

By repeated use of the triangle inequality, we can write

$$\|f(x) - 2^n f(2^{-n}x)\| \leq \|f(x) - 2f(2^{-1}x)\| + \cdots + 2^{n-1}\|f(2^{-n+1}x) - 2f(2^{-n}x)\|, \tag{5.4}$$

but by (5.3) we have

$$\begin{aligned} \|f(x) - 2f(2^{-1}x)\| + \cdots &+ 2^{n-1}\|f(2^{-n+1}x) - 2f(2^{-n}x)\| \\ &\leq 2^{1-p}\theta\|x\|^p + \cdots + 2^{n-1}2^{1-p}\theta\|2^{-n+1}x\|^p. \end{aligned} \tag{5.5}$$

Now, the right-hand side of (5.5) is a finite geometric series; i.e., it is given by $\sum_{i=1}^{n} 2^{i(1-p)}\theta\|x\|^p$. This finite sum is of course less than or equal to the infinite geometric series of the same form which converges to $\frac{2\theta\|x\|^p}{2^p-2}$. With this idea established, we will now proceed to prove Gajda's theorem which was presented in [8].

Theorem 5.2 *Let E_1 be a normed linear space and let E_2 be a Banach space. Let $f : E_1 \to E_2$ be a mapping for which there exist two constants $\theta \in [0, \infty)$ and $p > 1$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{5.6}$$

for all x, y in E_1. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta\|x\|^p}{2^p - 2} \tag{5.7}$$

for every x in E_1.

PROOF. Replacing x and y by $2^{-1}x$ in (5.6) gives

$$\|f(x) - 2f(2^{-1}x)\| \leq 2^{1-p}\theta\|x\|^p. \tag{5.8}$$

We claim that

$$\|f(x) - 2^n f(2^{-n}x)\| \leq \sum_{i=1}^{n} 2^{i(1-p)}\theta\|x\|^p \tag{5.9}$$

for any positive integer n. Clearly this is true for the case $n = 1$ since replacing n by 1 in (5.9) gives (5.8). We now replace x by $2^{-1}x$ in (5.9) and multiply the inequality by 2 to get

$$\|2f(2^{-1}x) - 2^{n+1}f(2^{-n-1}x)\| \leq \sum_{i=2}^{n+1} 2^{i(1-p)}\theta\|x\|^p. \tag{5.10}$$

Adding (5.8) and (5.10) by the triangle inequality yields

$$\|f(x) - 2^{n+1}f(2^{-n-1}x)\| \leq \sum_{i=1}^{n+1} 2^{i(1-p)}\theta\|x\|^p \tag{5.11}$$

so that the inductive assumption (5.9) is indeed true for any positive integer n. In fact, we can deduce from (5.9) that

$$\|f(x) - 2^n f(2^{-n}x)\| \leq \frac{2\theta\|x\|^p}{2^p - 2} \tag{5.12}$$

since the right-hand side of (5.9) is a geometric series.

Now, if m is a positive integer such that $m > n$, then $m - n \in \mathbb{N}$, and so (5.12) is equivalent to the following:

$$\|f(x) - 2^{m-n}f(2^{n-m}x)\| \leq \frac{2\theta\|x\|^p}{2^p - 2}. \tag{5.13}$$

If we replace x by $2^{-n}x$ in (5.13) we then get

$$\|f(2^{-n}x) - 2^{m-n}f(2^{-m}x)\| \leq 2^{-np}\frac{2\theta\|x\|^p}{2^p - 2} \tag{5.14}$$

which when multiplied through by 2^n yields

$$\|2^n f(2^{-n}x) - 2^m f(2^{-m}x)\| \leq 2^{n(1-p)}\frac{2\theta\|x\|^p}{2^p - 2}. \tag{5.15}$$

Since $p > 1$, the right-hand side of (5.15) converges to zero as n tends toward infinity for every fixed x in E_1; therefore, the sequence $\{2^n f(2^{-n}x)\}_{n=1}^{\infty}$ is a Cauchy sequence and converges to some point in E_2, because of the completeness of E_2. We call $T(x)$ the limit of this sequence; that is,

$$T(x) \equiv \lim_{n\to\infty} 2^n f(2^{-n}x),$$

for all $x \in E_1$.

Replacing x and y by $2^{-n}x$ and $2^{-n}y$, respectively, in (5.6) and multiplying the entire inequality through by 2^n gives

$$\|2^n f\left(2^{-n}(x+y)\right) - 2^n f(2^{-n}x) - 2^n f(2^{-n}y)\| \leq 2^{n(1-p)}\theta(\|x\|^p + \|y\|^p) \tag{5.16}$$

for each fixed x, y in E_1. Taking the limit of both sides of (5.16) as n tends to infinity yields

$$\|T(x+y) - T(x) + T(y)\| = 0 \tag{5.17}$$

or

$$T(x+y) = T(x) + T(y) \tag{5.18}$$

so that T is indeed an additive function.

Taking the limit of both sides of (5.12) as n tends to infinity gives the inequality

$$\|f(x) - T(x)\| \le \frac{2\theta\|x\|^p}{2^p - 2}. \tag{5.7}$$

Now, suppose that there exists another additive function $t : E_1 \to E_2$ such that

$$\|f(x) - t(x)\| \le \frac{2\theta\|x\|^q}{2^q - 2} \tag{5.19}$$

for some $q > 1$. Then

$$\|T(x) - t(x)\| \le \frac{2\theta\|x\|^p}{2^p - 2} + \frac{2\theta\|x\|^q}{2^q - 2} \tag{5.20}$$

results from adding (5.7) and (5.19) by the triangle inequality. Now both T and t are additive functions, so that

$$\|T(x) - t(x)\| = \|nT(n^{-1}x) - nt(n^{-1}x)\| = n\|T(n^{-1}x) - t(n^{-1}x)\|, \tag{5.21}$$

but this allows us to write

$$\|T(x) - t(x)\| \le n\frac{2\theta\|n^{-1}x\|^p}{2^p - 2} + n\frac{2\theta\|n^{-1}x\|^q}{2^q - 2} \tag{5.22}$$

or

$$\|T(x) - t(x)\| \le n^{1-p}\frac{2\theta\|x\|^p}{2^p - 2} + n^{1-q}\frac{2\theta\|x\|^q}{2^q - 2}. \tag{5.23}$$

Since both p and q are greater than 1, taking the limit of both sides of (5.23) as n tends to infinity results in

$$\|T(x) - t(x)\| = 0 \tag{5.24}$$

so that $T(x) = t(x)$ for every $x \in E_1$; therefore, the additive mapping T is the unique additive mapping satisfying (5.7). □

Making only a minor, obvious change in Theorem 4.2 above also proves that the mapping T in Gajda's theorem is linear whenever the mapping $f(tx)$ is continuous in t for all $t \in \mathbb{R}$.

CHAPTER

SIX

New Results

Since Rassias and Gajda both succeeded in generalizing upon the theorems of D. H. Hyers, one could ask how much further such generalizations can go. The author along with B. R. Ebanks began looking at this question by examining the inequality $\|f(x+y) - f(x) - f(y)\| \leq G(x,y)$ with the idea of determining what assumptions would have to be placed upon the mapping G in order to guarantee the existence and uniqueness of an additive mapping T which f would approximate. In the late 1940s and early 1950s, D. G. Bourgin also worked with approximate homomorphisms and approximate isometries as had Ulam and Hyers (cf. [3]). Bourgin appears to have been the first to examine the general inequality $\|f(x+y) - f(x) - f(y)\| \leq G(x,y)$. Unfortunately, it appears that no proof of Bourgin's results with this inequality has ever been published.

At first, we deduced one theorem involving the above inequality which was a quite natural extension of Hyers' Theorem (Theorem 3.1); however, it only included Rassias' inequality (given in Theorem 4.1) for the case of $p < 0$. B. Ebanks modified our inductive hypothesis so that our theorem was able to completely subsume the theorems of Hyers, Rassias, and Gajda. Our former theorem was retained as a corollary of this newer modified one. We give our new theorems and their corollaries here.

Theorem 6.1 *Let E_1 be a linear space and E_2 be a Banach space. Let $f : E_1 \to E_2$ and $G : E_1 \times E_1 \to [0, \infty)$ be mappings which together satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq G(x,y) \tag{6.1}$$

for all $x, y \in E_1$. Also, suppose G satisfies the following conditions for all x, y in E_1:

(i) $\sum_{i=n}^{m-1} G(2^i x, 2^i x) \cdot 2^{-i-1} \to 0$ *as $m, n \to \infty$, and*
(ii) $\lim_{n\to\infty} 2^{-n} G(2^n x, 2^n y) = 0$.

Then the formula $T(x) \equiv \lim_{n\to\infty} 2^{-n} f(2^n x)$ *defines an additive mapping* $T : E_1 \to E_2$ *such that*

$$\|T(x) - f(x)\| \le \sum_{i=0}^{\infty} G(2^i x, 2^i x) \cdot 2^{-i-1}. \tag{6.2}$$

Furthermore, if $2^{-n} \sum_{i=0}^{\infty} G(2^{i+n}x, 2^{i+n}x) \cdot 2^{-i}$ *converges to zero as* n *tends toward infinity, then* T *is the unique such additive mapping.*

Proof. Replacing y by x in (6.1) gives

$$\|f(2x) - 2f(x)\| \le G(x, x). \tag{6.3}$$

Next, we prove that

$$\|f(2^n x) - 2^n f(x)\| \le \sum_{i=0}^{n-1} G(2^i x, 2^i x) \cdot 2^{n-i-1} \tag{6.4}$$

for any positive integer n. Clearly (6.4) is true for the case $n = 1$, since setting $n = 1$ in (6.4) gives (6.3). Now, replacing x by $2^n x$ in (6.3) gives the following:

$$\|f(2^{n+1}x) - 2f(2^n x)\| \le G(2^n x, 2^n x). \tag{6.5}$$

Then we multiply (6.4) by 2 to get

$$\|2f(2^n x) - 2^{n+1} f(x)\| \le \sum_{i=0}^{n-1} G(2^i x, 2^i x) \cdot 2^{n-i}. \tag{6.6}$$

Adding (6.5) and (6.6) by the triangle inequality yields

$$\|f(2^{n+1}x) - 2^{n+1} f(x)\| \le \sum_{i=0}^{n} G(2^i x, 2^i x) \cdot 2^{n-i} \tag{6.7}$$

so that the inductive assumption (6.4) is indeed true for all positive integers.

We now wish to construct a Cauchy sequence of function values. First, we multiply (6.4) by 2^{-n} to get

$$\|2^{-n} f(2^n x) - f(x)\| \le \sum_{i=0}^{n-1} G(2^i x, 2^i x) \cdot 2^{-i-1}. \tag{6.8}$$

If $m > n > 0$, then $m - n \in \mathbb{N}$. Replacing n by $m - n$ in (6.8) gives

$$\|2^{n-m} f(2^{m-n} x) - f(x)\| \le \sum_{i=0}^{m-n-1} G(2^i x, 2^i x) \cdot 2^{-i-1}. \tag{6.9}$$

Then we replace x by $2^n x$ in (6.9) to get

$$\|2^{n-m} f(2^m x) - f(2^n x)\| \le \sum_{i=0}^{m-n-1} G(2^{i+n}x, 2^{i+n}x) \cdot 2^{-i-1}. \tag{6.10}$$

Dividing (6.10) by 2^n yields

$$\|2^{-m} f(2^m x) - 2^{-n} f(2^n x)\| \le \sum_{i=0}^{m-n-1} G(2^{i+n}x, 2^{i+n}x) \cdot 2^{-n-i-1}. \tag{6.11}$$

Now, by replacing i by $i-n$, it is clear that

$$\sum_{i=0}^{m-n-1} G(2^{i+n}x, 2^{i+n}x) \cdot 2^{-n-i-1} = \sum_{i=n}^{m-1} G(2^i x, 2^i x) \cdot 2^{-i-1},$$

which by supposition (i) converges to zero as m and n tend to infinity. Thus (6.11) shows that the sequence $\{2^{-n} f(2^n x)\}_{n=1}^{\infty}$ is a Cauchy sequence for each fixed $x \in E_1$. Since E_2 is a Banach space, then this sequence converges for each fixed x in E_1 to some limit in E_2. We call this limit $T(x)$.

Next, it is demonstrated that $T(x) \equiv \lim_{n\to\infty} 2^{-n} f(2^n x)$ defines an additive function. Replacing x and y by $2^n x$ and $2^n y$, respectively, in (6.1) and multiplying the inequality through by 2^{-n} yields

$$\|2^{-n} f\left(2^n(x+y)\right) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y)\| \le 2^{-n} G(2^n x, 2^n y). \tag{6.12}$$

By supposition (ii), the right-hand side of (6.12) goes to zero as n tends to infinity, so that taking the limit of (6.12) as n tends to infinity yields

$$\|T(x+y) - T(x) - T(y)\| = 0 \tag{6.13}$$

or

$$T(x+y) = T(x) + T(y). \tag{6.14}$$

Hence T is indeed an additive function.

From (6.8) we get

$$\lim_{n\to\infty} \|2^{-n} f(2^n x) - f(x)\| \le \lim_{n\to\infty} \sum_{i=0}^{n-1} G(2^i x, 2^i x) \cdot 2^{-i-1} \tag{6.15}$$

or

$$\|T(x) - f(x)\| \le \sum_{i=0}^{\infty} G(2^i x, 2^i x) \cdot 2^{-i-1}. \tag{6.2}$$

Since $\sum_{i=n}^{m-1} G(2^i x, 2^i x) \cdot 2^{-i-1} \to 0$ as $m, n \to \infty$ (by supposition (i)), the right-hand side of (6.2) converges.

Suppose that there exists another additive mapping $t : E_1 \to E_2$ such that

$$\|t(x) - f(x)\| \leq \sum_{i=0}^{\infty} G(2^i x, 2^i x) \cdot 2^{-i-1}. \tag{6.16}$$

Then (6.2) and (6.16) yield

$$\begin{aligned} \|T(x) - t(x)\| &= \|2^{-n}T(2^n x) - 2^{-n}t(2^n x)\| = 2^{-n}\|T(2^n x) - t(2^n x)\| \\ &\leq 2^{-n+1} \sum_{i=0}^{\infty} G(2^{i+n}x, 2^{i+n}x) \cdot 2^{-i-1} \end{aligned} \tag{6.17}$$

by the triangle inequality. If the right-hand side of (6.17) converges to zero as n tends to infinity, then one gets that $\lim_{n\to\infty} \|T(x) - t(x)\| = 0$ or $T(x) = t(x)$ for all x in E_1, so that T is the unique additive mapping satisfying (6.2). □

Rassias' majorizing function $\theta(\|x\|^p + \|y\|^p)$ given in (4.1) satisfies all three conditions for the mapping G as stated in Theorem 6.1; therefore, Theorem 4.1 follows as a corollary of Theorem 6.1. We make a note here that the linearity of the additive mapping T follows from the additional condition in Rassias' Theorem 4.2, which applies directly to our problem since we have defined the mapping $T \equiv \lim_{n\to\infty} 2^{-n} f(2^n x)$ in the same manner as did Rassias.

With the above theorem established, we can now verify a corollary which follows immediately. This corollary was originally proved as a separate theorem and as such was the starting point for our attempts to generalize upon the work of Hyers, Rassias, and Gajda. It can be verified by the reader that the following corollary contains Rassias' theorem (Theorem 4.1) for the case of $p < 0$.

Corollary 6.2 *Let E_1 be a linear space and E_2 be a Banach space. Let $f : E_1 \to E_2$ and $G : E_1 \times E_1 \to [0, \infty)$ be mappings which together satisfy inequality (6.1) for all x and y in E_1. Also, suppose G satisfies the additional conditions:*

(i) $G(2x, 2x) \leq G(x, x)$,
(ii) $\lim_{n\to\infty} 2^{-n} G(2^n x, 2^n y) = 0, \qquad \forall x, y \in E_1$.

Then there exists a unique additive mapping $T : E_1 \to E_2$ satisfying (6.2).

Since $G(2x, 2x) \leq G(x, x)$, then it follows immediately that

$$\sum_{i=n}^{m-1} G(2^i x, 2^i x) \cdot 2^{-i-1} \quad \text{and} \quad 2^{-n} \sum_{i=0}^{\infty} G(2^{i+n}x, 2^{i+n}x) \cdot 2^{-i}$$

both converge to zero as m and n tend to infinity. Also, condition (ii) of this corollary is identical to condition (ii) of Theorem 6.1. All conditions imposed upon G by the above theorem are satisfied so that the result of the theorem establishes the corollary.

The above theorem and corollary are analogous to Theorem 4.1. Now, we use a method introduced by Gajda in [8] to prove the analogue of Theorem 5.2, where the additive mapping T is defined by $T(x) \equiv \lim_{n\to\infty} 2^n f(2^{-n}x)$. In order to do this the inductive assumption of the previous theorem has to be modified. We shall set up our induction in the fashion used by Gajda. Again, we let f and G satisfy

$$\|f(x+y) - f(x) - f(y)\| \le G(x,y). \tag{6.1}$$

Replacing y by x in (6.1) gives

$$\|f(2x) - 2f(x)\| \le G(x,x). \tag{6.18}$$

Next, we replace x by $\frac{x}{2}$ in (6.18) to get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le G\left(\frac{x}{2}, \frac{x}{2}\right). \tag{6.19}$$

Again replacing x by $\frac{x}{2}$ in (6.19) gives

$$\left\|f\left(\frac{x}{2}\right) - 2f\left(\frac{x}{4}\right)\right\| \le G\left(\frac{x}{4}, \frac{x}{4}\right). \tag{6.20}$$

Continuing in this fashion and adding appropriate multiples of each of these inequalities by the triangle inequality, we arrive at the following:

$$\begin{aligned}\|f(x) - 2^n f(2^{-n}x)\| \quad &\le \quad \|f(x) - 2f(2^{-1}x)\| + 2\|f(2^{-1}x) - 2f(2^{-2}x)\| + \cdots + \\ &\quad + 2^{n-1}\|f(2^{-n+1}x) - 2f(2^{-n}x)\|.\end{aligned} \tag{6.21}$$

Now, the right-hand side of (6.21) is bounded above by

$$G(2^{-1}x, 2^{-1}x) + \cdots + 2^{n-1}G(2^{-n}x, 2^{-n}x) \le \sum_{i=1}^{n} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1}, \tag{6.22}$$

so that (6.21) implies

$$\|f(x) - 2^n f(2^{-n}x)\| \le \sum_{i=1}^{n} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1}. \tag{6.23}$$

With this observation, we are now ready to prove the following theorem.

Theorem 6.3 *Let E_1 be a linear space and E_2 be a Banach space. Let $f : E_1 \to E_2$ and $G : E_1 \times E_1 \to [0,\infty)$ be mappings which together satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le G(x,y) \tag{6.1}$$

for all $x, y \in E_1$. Also, suppose that G satisfies the following conditions for all $x, y \in E_1$:

(i) $\sum_{i=n+1}^{m} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1} \to 0$ *as $m, n \to \infty$, and*
(ii) $\lim_{n\to\infty} 2^n G(2^{-n}x, 2^{-n}y) = 0.$

Then, the formula $T(x) \equiv \lim_{n\to\infty} 2^n f(2^{-n}x)$ *defines an additive mapping* $T : E_1 \to E_2$ *such that*

$$\|T(x) - f(x)\| \leq \sum_{i=1}^{\infty} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1}. \tag{6.24}$$

Moreover, if $2^n \sum_{i=1}^{\infty} G(2^{-n-i}x, 2^{-n-i}x) \cdot 2^i \to 0$ *as* $n \to \infty$*, then* T *is the unique such additive mapping.*

Proof. Replacing y by x in (6.1) gives

$$\|f(2x) - 2f(x)\| \leq G(x, x). \tag{6.25}$$

Now replacing x by $2^{-1}x$ in (6.25) yields

$$\|f(x) - 2f(2^{-1}x)\| \leq G(2^{-1}x, 2^{-1}x) \tag{6.26}$$

We now claim that

$$\|f(x) - 2^n f(2^{-n}x)\| \leq \sum_{i=1}^{n} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1} \tag{6.27}$$

for all positive integers n. Clearly it is true for the case $n = 1$ since setting n equal to 1 in (6.27) gives (6.26). Replacing x by $2^{-1}x$ in (6.27) and multiplying the inequality through by 2 yields

$$\|2f(2^{-1}x) - 2^{n+1} f(2^{-n-1}x)\| \leq \sum_{i=1}^{n} G(2^{-i-1}x, 2^{-i-1}x) \cdot 2^i. \tag{6.28}$$

Adding (6.26) and (6.28) by the triangle inequality gives

$$\|f(x) - 2^{n+1} f(2^{-n-1}x)\| \leq \sum_{i=1}^{n+1} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1} \tag{6.29}$$

so that the inductive assumption (6.27) is true for all positive integers.

If $m \in \mathbb{N}$ such that $m > n > 0$, then $m - n \in \mathbb{N}$ so that n can be replaced by $m - n$ in (6.27) to yield the following inequality:

$$\|f(x) - 2^{m-n} f(2^{n-m}x)\| \leq \sum_{i=1}^{m-n} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1}. \tag{6.30}$$

Now, when we replace x by $2^{-n}x$ in (6.30) and then multiply the inequality through by 2^n, we get

$$\|2^n f(2^{-n}x) - 2^m f(2^{-m}x)\| \leq \sum_{i=1}^{m-n} G(2^{-i-n}x, 2^{-i-n}x) \cdot 2^{i+n-1}. \tag{6.31}$$

Replacing $i + n$ by i in (6.31) then gives

$$\|2^n f(2^{-n}x) - 2^m f(2^{-m}x)\| \leq \sum_{i=n+1}^{m} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1}. \tag{6.32}$$

By supposition (i), the right-hand side of (6.32) converges to zero as m and n both tend to infinity. This implies that the sequence $\{2^n f(2^{-n}x)\}_{n=1}^{\infty}$ is a Cauchy sequence. Since E_2 is a Banach space, then this sequence has a limit in E_2 for each fixed $x \in E_1$. We call this limit $T(x)$.

We now shall show that T as defined above is an additive mapping. Replacing x and y by $2^{-n}x$ and $2^{-n}y$, respectively, in (6.1) and multiplying the entire inequality by 2^n yields

$$\|2^n f\left(2^{-n}(x+y)\right) - 2^n f(2^{-n}x) - 2^n f(2^{-n}y)\| \leq 2^n G(2^{-n}x, 2^{-n}y). \tag{6.33}$$

Taking the limit of (6.33) as n tends toward infinity, we obtain

$$\|T(x+y) - T(x) - T(y)\| = 0 \tag{6.34}$$

or

$$T(x+y) = T(x) + T(y) \tag{6.35}$$

since by supposition (ii) the sequence $\{2^n G(2^{-n}x, 2^{-n}y)\}_{n=1}^{\infty}$ converges to zero. Thus, it is established that T is additive.

Now, it is demonstrated that under an additional hypothesis T is the unique such additive mapping. Taking the limit of (6.27) as n tends toward infinity gives

$$\|f(x) - T(x)\| \leq \sum_{i=1}^{\infty} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1}. \tag{6.24}$$

Suppose that there exists another additive mapping $t : E_1 \to E_2$ such that

$$\|f(x) - t(x)\| \leq \sum_{i=1}^{\infty} G(2^{-i}x, 2^{-i}x) \cdot 2^{i-1}. \tag{6.36}$$

Now, replacing x by $2^{-n}x$ in (6.24) and (6.36), multiplying both inequalities by 2^n, and then adding them by the triangle inequality yields

$$\|T(x) - t(x)\| = 2^n \|T(2^{-n}x) - t(2^{-n}x)\| \leq 2^n \sum_{i=1}^{\infty} G(2^{-n-i}x, 2^{-n-i}x) \cdot 2^i. \tag{6.37}$$

By supposition, the right-hand side of (6.37) goes to zero as n tends toward infinity; therefore, the additive mapping T is the unique additive mapping satisfying (6.24). □

As with Theorem 6.1, the linearity of the mapping T can again be handled by Rassias' Theorem 4.2 with the modifications required because of the alternative definition for the mapping T.

A theorem proved by Hyers and Rassias in [11] follows as a corollary of the above two theorems.

Corollary 6.4 *Let $G(s,t)$ be nonnegative for all nonnegative real numbers s and t and positive homogeneous of degree p where p is real and $p \neq 1$; i.e., $G(\lambda s, \lambda t) = \lambda^p G(s,t)$ for $\lambda > 0$. Let E_1 be a linear space and E_2 be a Banach space. Also let $f : E_1 \to E_2$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq G(\|x\|, \|y\|) \tag{6.38}$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ satisfying the inequality

$$\|f(x) - T(x)\| \leq \frac{G(1,1)\|x\|^p}{|2 - 2^p|} \tag{6.39}$$

for all $x \in E_1$.

The above corollary follows since the homogeneity of G guarantees the convergence of the two series and the sequence given in the conditions of Theorem 6.1 and Theorem 6.3. We note also that (6.39) is precisely the result predicted by Theorem 6.1 for $p < 1$ or by Theorem 6.3 for $p > 1$. In the case of Theorem 6.1, (6.2) gives the distance between f and T as

$$\|T(x) - f(x)\| \leq \sum_{i=0}^{\infty} G(2^i x, 2^i x) \cdot 2^{-i-1}.$$

Now, since G is positive homogeneous of degree $p \neq 1$, then the right-hand side of the above equation becomes

$$\sum_{i=0}^{\infty} G(2^i x, 2^i x) \cdot 2^{-i-1} = \frac{1}{2}\|x\|^p G(1,1) \sum_{i=0}^{\infty} 2^{i(p-1)},$$

but this latter sum converges to $\frac{2}{2-2^p}$, so that the result of the above corollary follows immediately. The case for Theorem 6.3 is nearly identical; however, one must take care to note the difference in the limits of the infinite series in Theorems 6.1 and 6.3. In Theorem 6.1 we are summing from zero to infinity, but in Theorem 6.3 we are summing from 1 to infinity (cf. [7] for a summary on limits of geometric series).

Because of the more general nature of our Theorems 6.1 and 6.3, we can extend this corollary of Hyers and Rassias a little further into the following more general corollary.

Corollary 6.5 *Let $G(s,t)$ be nonnegative for all s and t in some vector space E_1. If G is positive homogeneous of degree p for $p \neq 1$ (i.e., if $G(\lambda s, \lambda t) = \lambda^p G(s,t)$), and if $f : E_1 \to E_2$ and $G : E_1 \times E_1 \to [0, \infty)$ together satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq G(x,y) \tag{6.1}$$

then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{G(1,1)\|x\|^p}{|2 - 2^p|} \tag{6.39}$$

for all $x \in E_1$.

So far, we have managed to pull into two theorems (Theorems 6.1 and 6.3) a great deal of work accumulated over fifty years of research into this particular question of stability. Naturally, there are numerous other questions which could be asked about the stability of this particular equation but which are not answered here. For example, Gajda, Rätz, and others have examined the question of the stability of the Cauchy equation between two semi-groups. According to D. Cenzer (cf. [5]), S. Ulam has mentioned that the problem for this particular functional equation remained open for compact metric groups; however, new results have been found in this area by Cenzer, de la Harpe, and Karoubi.

CHAPTER

SEVEN

The Case of $p = 1$ in Rassias' Inequality

Notice in Theorems 4.1 and 5.2 that the infinite sums involved were convergent if $p < 1$ (as in Theorem 4.1) or if $p > 1$ (as in Theorem 5.2). Even when we note the fact that Theorems 4.1 and 5.2 follow as corollaries of Theorems 6.1 and 6.3, respectively, we find that we could not guarantee the convergence of the infinite sums involved in our inequalities for the case of $p = 1$. In Rassias' and Gajda's theorems, letting $p = 1$ does not provide a strong enough bound on the Cauchy difference to guarantee the existence of an additive mapping T which our function f should approximate.

As Gajda has pointed out in [8], the case of $p = 1$ in Rassias' inequality (4.1) has not been covered by either Rassias or himself. This was not merely coincidental or the result of some oversight. Rather it was because neither Rassias' or Gajda's theorems could be extended to include this case. Note that in Theorems 4.1 and 5.2 we can replace the term $\frac{2\theta}{|2-2^p|}$ by a constant δ_p for any fixed value of $p \neq 1$. We could then write (4.2) as

$$\|f(x) - T(x)\| \leq \delta_p \|x\|^p \tag{7.1}$$

for any value of p except $p = 1$. For the case of $p = 1$, is it possible to find a constant δ and an additive mapping T such that (7.1) is true? Gajda provided an irrefutable counterexample which shows that there is a function f which satisfies the conditions of Theorems 4.1 and 5.2 for $p = 1$, but that there is no additive mapping T and no constant δ such that (7.1) is true. For a counterexample, Gajda constructed a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x + y) - f(x) - f(y)| \leq \theta(|x| + |y|) \tag{7.2}$$

for all $x, y \in \mathbb{R}$ and for every $\theta \in \mathbb{R}$. Were Theorem 4.1 applicable to this case, we would then expect the result that there existed some additive mapping $T : \mathbb{R} \to \mathbb{R}$ and some constant $\delta \geq 0$ such that

$$|f(x) - T(x)| \leq \delta|x| \tag{7.3}$$

for every real x. We demonstrate that this could not be the case by reproducing here Gajda's counterexample. We choose some $\theta > 0$ and then set $\mu = \frac{\theta}{6}$. Now, define a function $\phi : \mathbb{R} \to \mathbb{R}$ by the following:

$$\phi(x) = \begin{cases} \mu & \text{for} \quad x \in [1, \infty) \\ \mu x & \text{for} \quad x \in (-1, 1) \\ -\mu & \text{for} \quad x \in (\infty, -1] \end{cases} \tag{7.4}$$

Now, the function ϕ is continuous and $|\phi(x)| \leq \mu$ for all real x. A function $f : \mathbb{R} \to \mathbb{R}$ is now defined by

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x) \tag{7.5}$$

for all $x \in \mathbb{R}$. Note that $2^{-n}\phi(2^n x) \leq 2^{-n}\mu$ for all $x \in \mathbb{R}$ and for any positive integer n, and clearly the series $\sum_{n=0}^{\infty} 2^{-n}\mu$ converges to 2μ. Therefore, by the Weierstrass M-Test [9], the right-hand side of (7.5) is a uniformly convergent series of continous functions. The function f defined above is then itself continuous. Also, we notice that

$$|f(x)| \leq \sum_{n=0}^{\infty} 2^{-n}\mu \tag{7.6}$$

so that

$$|f(x)| \leq 2\mu \tag{7.7}$$

for all $x \in \mathbb{R}$.

Now, it is shown that the function f defined in (7.5) satisfies (7.2). First, we note that $f(0) = 0$ so that if $x = y = 0$ then (7.2) is certainly fulfilled. Next, we assume that $0 < |x| + |y| < 1$. Then there exists a positive integer m such that

$$\frac{1}{2^m} \leq |x| + |y| < \frac{1}{2^{m-1}}. \tag{7.8}$$

Therefore, $|2^{m-1}x| + |2^{m-1}y| < 1$ so that $|2^{m-1}x| < 1$, $|2^{m-1}y| < 1$, and $|2^{m-1}(x+y)| \leq 2^{m-1}(|x| + |y|) < 1$. For every positive integer $n \in \{0, 1, \ldots, m-1\}$ each of $2^n x$, $2^n y$, and $2^n(x+y)$ remain in the interval $(-1, 1)$, over which the function ϕ is linear; therefore, it is true that

$$\phi\left(2^n(x+y)\right) - \phi(2^n x) - \phi(2^n y) = 0 \tag{7.9}$$

for all integers n between 0 and $m-1$, inclusive. From (7.9) and (7.5) we get

$$\sum_{n=0}^{\infty} \frac{|\phi\left(2^n(x+y)\right) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|)} \leq \sum_{n=m}^{\infty} \frac{|\phi\left(2^n(x+y)\right) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|)}. \tag{7.10}$$

Because $|\phi(x)| \leq \mu$ for all $x \in \mathbb{R}$, we then have that $|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)| \leq 3\mu$. Now we know that

$$\sum_{n=m}^{\infty} \frac{|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x|+|y|)} \leq 3\mu \sum_{n=m}^{\infty} \frac{1}{2^n(|x|+|y|)}, \tag{7.11}$$

but by (7.8) $1 \leq 2^m(|x|+|y|)$. This implies then that the right-hand side of (7.11) converges to something less than $6\mu = \theta$. From this, we then have

$$\frac{|f(x+y) - f(x) - f(y)|}{|x|+|y|} \leq \theta \tag{7.12}$$

so that the function f as defined by (7.5) does indeed satisfy (7.2) for $0 \leq |x|+|y| < 1$.

For the case of $|x|+|y| \geq 1$, the mapping f defined by (7.5) also satisfies (7.2). By (7.5), we can write

$$\begin{aligned} |f(x+y) \quad - \quad & f(x) - f(y)| = \\ = \quad & \left| \sum_{n=0}^{\infty} 2^{-n}\phi(2^n(x+y)) - \sum_{n=0}^{\infty} 2^{-n}\phi(2^n x) - \sum_{n=0}^{\infty} 2^{-n}\phi(2^n y) \right|. \end{aligned} \tag{7.13}$$

For all $x \in \mathbb{R}$, we have that $|\phi(x)| \leq \mu$, so that the right-hand side of (7.13) is less than or equal to $3\sum_{n=0}^{\infty} 2^{-n}\mu$, which converges to 6μ. Above we had chosen $\mu = \frac{\theta}{6}$, so that we can now write

$$|f(x+y) - f(x) - f(y)| \leq \theta, \tag{7.14}$$

which implies

$$|f(x+y) - f(x) - f(y)| \leq \theta(|x|+|y|) \tag{7.15}$$

since $|x|+|y| \geq 1$, and $\theta \leq \theta(|x|+|y|)$.

Now, suppose that there exists a real number $\delta \geq 0$ and an additive mapping $T : \mathbb{R} \to \mathbb{R}$ such that (7.3) is true. Because of the continuity of f, it follows that T is bounded on some open interval containing zero. As we discussed earlier in this paper, Darboux proved that, if T is a solution of Cauchy's equation, and T is bounded on any open interval, then T is of the form

$$T(x) = cx \tag{7.16}$$

for all real x and some real number c. Substituting (7.16) into (7.3) gives

$$|f(x) - cx| \leq \delta|x| \tag{7.17}$$

for all $x \in \mathbb{R}$. Now, (7.17) implies that

$$\left|\frac{f(x)}{x}\right| \leq \delta + |c| \tag{7.18}$$

for all $x \in \mathbb{R}$.

There is, however, an integer m large enough that $m\mu > \delta + |c|$. Now, if $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0, 1)$ for each integer $n \in \{0, 1, \ldots, m-1\}$. Then, for such a value x, we have

$$\frac{f(x)}{x} \geq \sum_{n=0}^{m-1} \frac{\phi(2^n x)}{2^n x} = \sum_{n=0}^{m-1} \frac{\mu 2^n x}{2^n x} = m\mu > \delta + |c|. \tag{7.19}$$

But this contradicts (7.18); therefore, Theorems 4.1 or 5.2 do not extend to the case $p = 1$.

We note also that Rassias' majorizing function given in (5.6) for the case $p = 1$ does not fit the complete set of conditions of either Theorem 6.1 or Theorem 6.3; therefore, we cannot guarantee the existence of an additive mapping T approximated by a function f even when inequality (6.1) is satisfied.

CHAPTER

EIGHT

Some Further Observations

The usefulness of our two new theorems lies in the fact that they do not depend upon the homogeneity of the function $G(x,y)$ which majorizes the Cauchy norm in the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq G(x,y). \tag{6.1}$$

We can find functions G which are not homogeneous but which satisfy the conditions of either Theorem 6.1 or Theorem 6.3. One such function is used in the following inequality:

$$|f(x+y) - f(x) - f(y)| \leq (|x| + |y|)^2 e^{|x|+|y|}, \tag{8.1}$$

where the mappings $f : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, defined by $G(x,y) = (|x| + |y|)^2 e^{|x|+|y|}$, satisfy (8.1) for all $x, y \in \mathbb{R}$. It is easily verified that the mapping G as defined above satisfies all three conditions given in Theorem 6.3; however, G is not a homogeneous function. The mapping G in this example is also unbounded so that the above inequality does not fit the criterion of Hyers' theorem, but it does lie "close" to the additive mapping T defined in Theorem 6.3.

We can also give another function $G : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ which is not homogeneous and which is not bounded but which satisfies the conditions of Theorem 6.1 and, therefore, lies "close" to some additive mapping. Take the mapping G defined by

$$G(x,y) = \begin{cases} \frac{|\sin x|}{|x|+|y|}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases} \tag{8.2}$$

which would give us the inequality

$$|f(x+y) - f(x) - f(y)| \leq \frac{|\sin x|}{|x| + |y|}, \tag{8.3}$$

when $(x,y) \neq (0,0)$. Theorem 6.1 guarantees the existence of an additive mapping $T : \mathbb{R} \to \mathbb{R}$ which would be approximated by any mapping $f : \mathbb{R} \to \mathbb{R}$ which satisfies inequality (8.3).

Since D. Hyers presented his solution to Ulam's question of the stability of Cauchy's fundamental functional equation in 1941, we have seen several new ideas in how to define stability. We have progressed from requiring the boundedness of the norm of the Cauchy difference to requiring that the Cauchy norm be majorizable by a particular homogeneous function. Now, we have managed to require only that the Cauchy norm be majorizable by any function (homogeneous or not) which satisfies either of two sets of three conditions. Admittedly, the three conditions in the statements of Theorems 6.1 and 6.3 are not aesthetically "nice," nor perhaps are they always easy to verify for some majorizing functions.

The author must of course ask now how one might go about generalizing upon the new results obtained here. Perhaps this could be done by looking at the above theorems in more general spaces. We have worked here entirely in normed linear spaces, but there are certainly more general spaces for which one might also obtain positive results in this particular problem of stability.

BIBLIOGRAPHY

[1] J. Aczél, *Lectures on Functional Equations and their Applications.* New York: Academic Press, 1966.

[2] J. Aczél, *A Short Course on Functional Equations: Based Upon Recent Applications to the Social and Behavioral Sciences.* Dordrecht: D. Reidel Publishing Co., 1987.

[3] D. G. Bourgin, "Classes of Transformations and Bordering Transformations," *Bulletin of the American Mathematical Society* **57** (1951), pp. 223-237.

[4] E. Castillo and M. R. Ruiz-Cobo, *Functional Equations and Modelling in Science and Engineering.* New York: Marcel Dekker, 1992.

[5] D. Cenzer, "The Stability Problem: New Results and Counterexamples," *Letters in Mathematical Physics* **10** (1985), p. 155-160.

[6] L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations.* New York: Academic Press Inc., Publishers, 1963.

[7] R. Ellis and D. Gulick, *Calculus with Analytic Geometry*, 3rd ed. San Diego: Harcourt Brace Jovanovich, Publishers, 1986.

[8] Z. Gajda, "On Stability of Additive Mappings," *International Journal of Mathematics and Mathematical Science* **14** (1991), pp. 431-434.

[9] E. Gaughan, *Introduction to Analysis*, 4th ed. Pacific Grove, CA: Brooks/Cole Publishing Company, 1993.

[10] D. Hyers, "On the Stability of the Linear Functional Equation," *Proceedings of the National Academy of Science of the U.S.A.* **27** (1941), pp. 222-224.

[11] D. H. Hyers and T. M. Rassias, "Approximate Homomorphisms," *Aequationes Mathematicae* **44** (1992), pp. 125-153.

[12] T. M. Rassias, "On the Stability of the Linear Mapping in Banach Spaces," *Proceedings of the American Mathematical Society* **72** (1978), pp. 297-300.

[13] J. Rätz, "On Approximately Additive Mappings." In: *General Inequalities 2.* Basel: Birkhäuser, 1980, pp. 233-251.

[14] J. Smital, *On Functions and Functional Equations.* Bristol: Adam Hilger, 1988.

[15] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed. New York: John Wiley & Sons, 1980.

[16] S. M. Ulam, *A Collection of Mathematical Problems.* New York: Interscience Publ., 1960.