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Research Statement

Decay of correlations

In 1998, Lai-Sang Young (See [You98].) published a "grand unified theory" (GUT) of exponential decay of correlations for dynamical systems which can be recognized as special factor systems of an "expanding" Kakutani-type tower system. (See [Kak43].) One of the crucial assumptions in this theory is that the unstable distortion between any two points is always uniformly bounded. In my Ph.D. thesis, I show that Young's GUT can be extended under appropriate assumptions to include dynamical systems for which this distortion grows without bound. In particular, a subset Λ of the the phase space is decomposed into countably many subsets $\{\Lambda_i\}_{i\in\mathbb{N}}$ according to the return time function (in the sense of Kakutani) and depending upon the singularity set of the system, and the distortion on Λ_i is bounded above by $C(i)$. If $\varrho(i)$ is the measure of Λ_i, then we require that the sum $\sum_{i\in\mathbb{N}} C(i)e^{C(i)}\varrho(i)$ converge.

This extension allows us to expand the collection of dynamical systems which fit into the GUT. In particular some very simple one-dimensional dynamical systems are easily shown to fit into our extended GUT but not into the original. One dynamical system of particular interest is a piecewise hyperbolic system constructed by S. Newhouse and M. Jakobson in [JN00] in which the unstable distortion between any two points in the same stable leaf has no uniform bound independent of the stable leaf. One would like to have reasonable geometric assumptions which would permit the growth of this distortion to be regulated by the decay of the measures of the sets on which the distortion grows larger. This problem is interesting because Newhouse and Jakobson show how it yields important new information about Hénon attractors, a phenomenon of great interest to astronomers and mathematicians since the 1970s.

Polynomial decay by the stochastic coupling method

In a subsequent paper [You99], Young obtains polynomial decay of correlations for many systems by applying the stochastic coupling method to the tower. The essential ingredient in this method and in the approach above is the (uniform) Hölder continuity of the jacobian of the dynamical system. Uniformly bounded distortion yields uniform Hölder continuity of the jacobian, but quickly growing distortion yields quick worsening of this Hölder continuity; thus, it should be true but must be verified that the extension we obtained in Young's original GUT should work here. This would be useful for instance since Alves, Luzzatto, and Pinheiro in [ALP] use this latter result directly to show that the rate of decay of correlations can be determined by how quickly "typical" points begin to exhibit exponential growth of the derivative as determined by the Lyapunov exponents. To apply the result of [You99], Alves, Luzzatto, and Pinheiro require uniformly bounded distortion—something with which we could dispense under our desired extension.

Stochastic stability

We say that a dynamical system (f, μ) is **stochastically stable** if there is some ϵ-neighborhood U_ϵ of f (in some appropriate metric; e.g., C^0, C^1, Hölder, etc.) such that there is a stationary probability measure μ_ϵ for all infinite random compositions $\cdots \circ f_j \circ f_{j-1} \circ \cdots \circ f_1$ of functions $f_i \in U_\epsilon$ such that $\int \varphi \, d\mu_\epsilon \to \int \varphi \, d\mu$ as $\epsilon \to 0$ for all continuous observables φ. One general rule of thumb is that systems which enjoy exponential decay of correlations tend to be stochastically stable. This is because the contraction of the Perron-Frobenius operator $\mathcal{L}$ often improves averages of the perturbed operator $\mathcal{L}_\epsilon$, although this is not usually sufficient to prove stochastic stability.

Stochastic stability is the probabilistic counterpart of structural stability. If we would like to use f to model some physical system, we cannot be sure that our function accommodates all physical phenomena present, but we would like the values predicted by f to be within some suitably small ϵ of the experimentally observed values. In other words, the experimental value is the value predicted by some function $f_1 \in U_\epsilon$. Plugging this back into the system, the error in f means that the new value is the value predicted by some function $f_2 \in U_\epsilon$, and so on. Structural stability would mean that all nearby systems would have orbit structures precisely like those of f, but this is usually too much to desire, and so we hope that the orbit structures are statistically like those of f.

Stochastic stability has not been proved for a wide collection of dynamical systems in general. Some specific types of dynamical systems have been shown to be stochastically stable; e.g., see [Via] and [Cow00]. I would like to examine the question of stochastic stability for the tower map in our current GUT. The question of stochastic stability for the original GUT is to my knowledge entirely unknown.

Perron-Frobenius operator

I am also interested in the general study of the Perron-Frobenius operator $\mathcal{L}$ itself. This operator and the properties of its spectrum are essential tools in the study of rates of mixing for dynamical systems, but the study of its spectrum yields other information. For example, Dellnitz, Froyland, and Sertl in [DFS00] show that certain isolated spectral points of $\mathcal{L}$ correspond to densities which converge to the invariant density very slowly. Also, Bratteli and Jorgensen in [BJ02] discuss the importance of $\mathcal{L}$ and its spectral properties in wavelet theory.

References

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