Data Analysis

An Introduction to Error Analysis and Data Presentation

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A.1 Average and Standard Deviation and Uncertainty

The accuracy of measurements can be improved upon by taking several readings and calculating an average, or *mean*, of the reading. The amount of fluctuations in the readings can be quantified by calculating the standard deviation of the readings.

A.1.1 Mean or Average

For N measurements of a quantity, the *mean* is given by,

(A1.1) $$\bar{x} \equiv \frac{\sum_{1}^{N} x_i}{N},$$

where the numerator is a sum from i=1 to i=N of the *x* measurements.

A.1.2 Standard Deviation

Repeated measurements of a quantity can vary randomly. (They can be non-random as well, but such variations will not be discussed here.) The statistical fluctuations can be quantified by calculating the *standard deviation*, which is given by,

(A.1.2) $$\sigma \equiv \sqrt{\frac{\sum_{1}^{N} (x_i - \bar{x})^2}{N-1}},$$

where x_i is the ith measurement.

Example A.1.1

If the measurements of a quantity were

18.6
19.3
17.7
20.4

The mean from equation (1.1) is $\bar{x} = 19.0$.

The standard deviation is,

$$\sigma = \sqrt{\frac{(18.6 - 19.0)^2 + (19.3 - 19.0)^2 + (17.7 - 19.0)^2 + (20.4 - 19.0)^2}{4 - 1}} = 1.14$$

(Probability theory predicts that for random fluctuations 68.3% of measurements will lie within 1 standard deviation of the mean.)

Example A.1.2

Table A.1.1 Example Data for Average and Standard Deviation			
Reading	length x (cm)	x_i -x_{avg}(cm)	(x_i -x_{avg})^2(cm)^2
1	2.33	0.05	0.0025
2	2.12	-0.16	0.0256
3	2.44	0.16	0.0256
4	2.14	-0.14	0.0196
5	2.35	0.07	0.0049
6	2.28	0	0
7	2.3	0.02	0.0004
average length	2.28	total $\Sigma(x_i-x_{avg})^2(cm)^2=$	0.0786
	Standard Deviation	$\sigma = \sqrt{\frac{0.0786}{7-1}}$	0.114 (cm)

A.1.3 Quoting Results with Uncertainties

All measurements have uncertainties associated with them. When reporting results, one must include the uncertainty. A standard format for this is,

(A.1.3) $\boxed{x \text{ +/- } \sigma}$

Uncertainties are determined a number of ways, some of which will be described below. The standard deviation, which was described above conveys one type of uncertainty, thus one can use it when one has many measurements of a value.

In example A.1.1 the result would be stated as

19.0 +/- 1.14

A.2 Uncertainties in Measurements and Significant Figures

With any measurement of a physical quantity there, is an uncertainty associated with the measurement. The sources of the uncertainties can be many and some can be difficult to discern. Uncertainties must be determined and reported when taking measurements in order to properly understand the significance of a result.

A.2.1 Least Count

One source of uncertainty is the level of precision of the measuring device. By convention, one considers the precision of an instrument to be one half of the *least count* of the instrument. For example, when using a ruler graded to mm, the precision is said to be 0.5mm. Every measuring device has a limit, or least count.

It is acceptable to estimate and report one digit beyond least count of the measuring device.

Example A.2.1
Suppose one uses a ruler graded to mm to measure the length of a piece of cloth and finds it to be between 3.7 cm and 3.8 cm. It next digit is estimated so that the measurement is given as 3.72 cm. The uncertainty due to the least count of the ruler is 0.5 mm or 0.05 cm), so that the resulting measurement should be stated as:

> 3.72 +/- 0.05 cm,

where the +/- 0.05 cm is due to the least count of the ruler

A.2.2 Other Sources of Uncertainty

There may be sources of uncertainty other than the least count of the measuring device. One must study the measurement procedure and think carefully about other causes of uncertainty. In the above example, suppose the cloth has frayed edges. The experimenter must make an estimate of how much uncertainty is introduced into the measurement and include that in the uncertainty.

To help in determining the magnitude of uncertainty one can ask and answer the following questions;

> Aside from the least count of the measuring apparatus, how well, or how precisely, do I know the value of what is being measured?

or to put it another way;

> Over what range can I alter my observation and not detect any difference in my reading?

Example A.2.2
If the length of the cloth ranges from 3.5 cm to 3.9 cm one should convey this uncertainty and state the measurement of length as length as:

3.7 +/- 0.2 cm.

where the +/- 0.2 cm is due to the fluctuations observed (and *not* due to the least count).

Example A.2.3
One may be trying to measure the location of where a ball landed by the mark the ball makes on a piece of paper. Although the ruler being used may be graded to millimeters, the mark made on the paper may just be a smudge that is larger than a millimeter. You must decide how precisely you know the location of the center of the smudge.

A.2.3 Reporting Uncertainties in Measurements

All measurements have uncertainties associated with them.

A good way to report uncertainties is to list them in a table.

Example A.2.3.1
Suppose we were measuring the position of an insect and had the following uncertainties:

C= 93 cm
σ_1 = 0.3 cm (due to odd shape of insect)
σ_2 = 0.4 cm (due to the bug wiggling around)
σ_3 = 0.05 cm (due to least count of ruler)

Table A.2.3.1 Table of Uncertainties

Variable Used for Measured Quantity	**Description of Measured Quantity**	**Uncertainty in Measurement or Measurements (cm)**	**Source of Uncertainty**
C	Position of insect	0.3	Due to odd shape of insect
C	Position of insect	0.4	Due to the insect wiggling around
C	Position of insect	0.05	Least count of ruler

A.2.4 Significant Figures

Since measurements have uncertainties associated with them, it follows that the quantities reported as a result of measurement for calculations done from measurements should include only enough decimal places to have meaning. The digits in a value that have meaning are referred to as **significant figures**.

A.2.4.1 Reporting Significant Figures

In the above example a ruler graded to mm was used to measure a length of a piece of cloth. The measurement was given as 3.72 +/- 0.05 cm. All three digits (3, 7, and 2), even the last, which was estimated, are significant, since they all have some degree of reliability. The number of significant figures should be apparent in how the value is reported. For example we would report the length in meters and kilometers as is shown in Table A.2.4.1.1

Table A.2.4.1.1 Length of Cloth With zeros to the left of the Significant Figures.
3.72 cm
0.0372 m
0.0000372 km

In each case there are only zeros to the left of the digits and this tells us that the zeros are not significant, but the digits are.

If we were to report this result is micro meters (10^{-6}m) and wrote it as 37200 μm there would be confusion as to whether or not the last two digits are significant. We know that in this example they are not. This is where scientific notation comes to the rescue. With scientific notation we can keep only the significant figures. Table A.2.4.1.2 shows the same result given in micrometers and nanometers.

Table A.2.4.1.2 Length of Cloth With zeros to the right of the Significant Figures.
3.72 cm
3.72×10^{-2} m
3.72×10^{4} μm
3.72×10^{7} nm

A.2.4.2 Calculations with Significant Figures.

Multiplication and Division
As a general rule the result should not have more significant figures than the least precise value used in the calculation. Here is an example.

Measure width: 4 cm
Measured length: 2.2 cm

Calculated Area: (4cm)(2.2cm) = 9cm

A.3 Independent Uncertainties for a Single Measurement

Uncertainties in measurements can come from more than one source. For example, when determining the position of an object, the least count of the ruler being used is one source of uncertainty. Another source could be that the object itself if wiggling around for some reason while the readings are being taken. Both factors will contribute to the total uncertainty of the reading.

Since the two factors are completely independent causes for uncertainty, the values of uncertainty do not simply add. Both causes of uncertainty are random (also called *statistical* or *indeterminate* error), thus, simply adding the two numbers results in an overestimate the total uncertainty.

To understand how to combine the uncertainties, we make an analogy with x, y and z variables in a rectangular Cartesian coordinate system. The coordinates, x, y and z are independent and a net displacement is given by Pythagorean's theorem:

(A.3.1) $$d^2 = x^2 + y^2 + z^2$$

or,

(A.3.2) $$d = \sqrt{x^2 + y^2 + z^2}$$

When making measurements, we call the uncertainty σ. Thus for a measurement with two or more uncertainties we have

(A.3.3) $$\sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + ...$$

or,

(A.3.4) $$\boxed{\sigma_{tot} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + ...}}$$

This is sometimes referred to as **adding the uncertainties "in quadrature"**.

The value is then reported as,

(A.3.5) $$\boxed{X \pm \sigma_{tot} = X \pm \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + ...}}$$

Example A.**3.1:**

Measure a position of an insect, C with two causes for uncertainty, σ_1 and σ_2.
C= 93 cm
σ_1 = 0.3 cm (due to odd shape of insect)
σ_2 = 0.4 cm (due to the bug wiggling around)

Total uncertainty is $\sigma_{tot} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \sqrt{(0.3cm)^2 + (0.4cm)^2} = 0.5000cm$
Final answer is C = 93 +/- 0.50 cm

Example A.**3.2:**
Often there are many sources of uncertainty and often only some of the sources contribute significantly to the total uncertainty. That is, some terms in the sum are small and can be ignored.

We repeat the above example, but now taking into account the uncertainty from a ruler graded to 1mm. We use half the least count as the uncertainty due to the ruler, or

σ_3=0.05 cm

Thus we have,

Measure a position of an insect, C with two causes for uncertainty, σ_1 and σ_2.
C= 93 cm
σ_1 = 0.3 cm (due to odd shape of insect)
σ_2 = 0.4 cm (due to the bug wiggling around)
σ_3 = 0.05 cm (due to least count of ruler)

Total uncertainty is

$$\sigma_{tot} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \sqrt{(0.3cm)^2 + (0.4cm)^2 + (0.05cm)^2} = 0.5025cm$$

Final answer is C = 93 +/- 0.5025 cm

Our previous result for the uncertainty was 0.5cm. Thus one can see that, in this case, one of the uncertainties, σ_3, has little effect on the total uncertainty since it is small compared to the other two uncertainties.

A.4 Differential Changes in Functions

We now wish to consider small changes in a function, called **differential changes**, *df*. That is, instead of considering uncertainties, we simply wish to have a functional relationship for small changes in the independent variables of a function.

Suppose we have a function that is a function of several variables,

A.4.1 $f=f(q_1,q_2,q_3,q_4,\ldots)$.

A general way to write the differential change in a function of several variables is,

A.4.2 $$\boxed{df = \frac{\partial f}{\partial q_1}dq_1 + \frac{\partial f}{\partial q_2}dq_2 + \frac{\partial f}{\partial q_3}dq_3 + \frac{\partial f}{\partial q_4}dq_4 + \ldots}$$

For example we write the differential change in a function in Cartesian coordinates as:

A.4.3 $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Example 1

$$f = x^2 + y^2 + z$$
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 2xdx + 2ydy + dz$$

This holds for other coordinate systems, for example spherical:

A.4.4 $$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \varphi}d\varphi$$

It also holds for a function of mixed variables, for example

Example 2

$$f = x^2 + y^2\cos(2\theta)$$
$$df = \frac{\partial\left(x^2 + y^2\cos(2\theta)\right)}{\partial x}dx + \frac{\partial\left(x^2 + y^2\cos(2\theta)\right)}{\partial y}dy + \frac{\partial\left(x^2 + y^2\cos(2\theta)\right)}{\partial \theta}d\theta$$
$$= 2xdx + 2y\cos(2\theta)dy - 2y^2\sin\theta d\theta$$

Here are more miscellaneous examples.

Example 3

$$f = x^2 y + y^2 + zy^3$$
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$
$$= \frac{\partial(x^2 y + y^2 + zy^3)}{\partial x}dx + \frac{\partial(x^2 y + y^2 + zy^3)}{\partial y}dy + \frac{\partial(x^2 y + y^2 + zy^3)}{\partial z}dz$$
$$= 2xydx + (x^2 + 2y + 3zy^2)dy + y^3 dz$$

Example 4

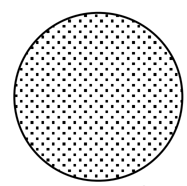
$$f = lnx$$

$$df = \frac{\partial(\ln x)}{\partial x}dx = \frac{dx}{x}$$

Here are a few examples with some geometrical interpretation, and which can be useful in setting up integrals.

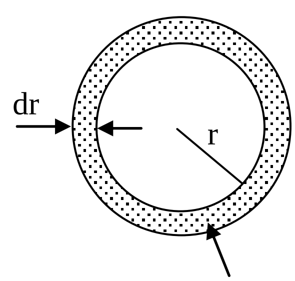
Example 5

$f = \pi r^2$ This is the equation for the area of a circle:



area=πr^2

If we take the differential of the area, we are varying the area, making it bigger by a little bit. Thus we are expanding the circle and the *expanded area* is an annulus.

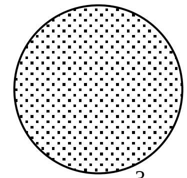


$$df = \frac{\partial(\pi r^2)}{\partial r}dr = 2\pi r dr$$

area of annulus = $2\pi r dr$

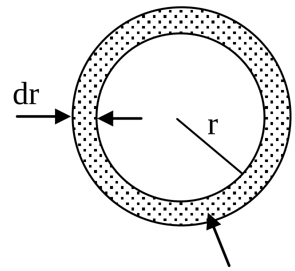
Example 6

$$f = \frac{4}{3}\pi r^3$$ This is the equation for the *volume* of a *sphere*:



volume of sphere=$4/3\pi r^3$

If we take the differential of the volume, we are varying the volume making it bigger by a little bit. Thus we are expanding the sphere and the *expanded volume* forme a spherical shell.



$$df = \frac{\partial\left(\frac{4}{3}\pi r^3\right)}{\partial r}dr = 4\pi r^2 dr$$

area of shell = $4\pi r^2$dr

Example 7

Here's an example from one dimensional kinematics.

We start with the equation relating position to velocity and time.

$$x = vt$$

Without assuming *v* or *t* are constant, we make a small change in position. That is, we take the differential:

$$dx = \frac{\partial(vt)}{\partial v}dv + \frac{\partial(vt)}{\partial t}dt$$

$$dx = tdv + vdt$$

To see what this means, we can identify the velocity, *v*, in our original equation, *x=vt*, as the velocity *before* we make the small change. With that in mind we re-label it, v_o.

$$dx = tdv_o + v_o dt$$

Next we divide both sides by *dt*.

$$\frac{dx}{dt} = \frac{tdv_o}{dt} + \frac{v_o dt}{dt}$$

The term on the left side of the equation is the velocity after the small change was made. We will define this as, $v=dx/dt$. The term, dv_o/dt, is the acceleration before the small change was made. We thus have,

$$v = ta + v_o$$

or in the more familiar form,

$$v = v_o + at$$

B Error Propagation

B.1 Introduction

Errors in individual variables must be accounted for when the variable is used in an equation. That is, an uncertainty in one variable affects other variables. For example one might be calculating position from the equation, $y=½at^2$. Uncertainty in measurements of time will result in uncertainties in the calculation of position. Accounting for this affect is called *error propagation.*

One can obtain a rough, but reasonable error propagation results by varying the measured values with their uncertainties. This method we will call the "max/min" method and it has the advantage of not needing calculus.

A more advanced method is to use the methods of differential calculus. This is especially useful for more complicated equations.

Both methods will now be described.

B.2 Error Propagation Using Max/Min Values

If one knows the uncertainty in a measure variable, a very straightforward method to estimate the uncertainty in a resulting function is to simply substitute the extremes high and low values into an equation.

The **advantages** of this method are that it is more straightforward to understand and sometimes carry out, compared to calculus methods.

The **disadvantages** are that it can involve more steps than calculus methods and, in general, it overestimates the uncertainties.

Example B.2.1 **f = A/B**

Given

$f = A/B$
$A = 5$
$B = 10$
$\sigma_A = 0.1$
$\sigma_B = 0.2$

We wish to find the uncertainty in the result, σ_f.

We have

$$f = 5/10 = 0.50$$

The largest value for f occurs when A is large and B is small,

$$f_{max} = (5 + 0.1)/(10-0.2) = 5.1/9.8 = 0.52$$

The smallest value for f occurs when A is small and B is large,

f_{min} = (5 - 0.1)/(10+0.2) = 4.9/10.2 = 0.48

We can then estimate σ_f by,

$$\sigma_f = \frac{|f_{max} - f_{min}|}{2}$$

In this example

σ_f = 0.02.

Our final answer would be

f = 0.50 +/- 0.02

Example B.2.2 f = lnA General Example

A = value
σ_A = uncertainty in A

We wish to find the uncertainty in f, σ_f.

We have,

A+σ = max value
A-σ = min value

$f = \ln(A)$
$f_{max} = \ln(A+\sigma)$
$f_{min} = \ln(A-\sigma)$

One can calculate the average of the high and low uncertainty as:

σ_{avg}= [(f_{max}-f)+(f-f_{min})]/2 = [(ln(A+σ)- ln(A))+(ln(A)- ln(A-σ))]/2

Example B.2.2 f = lnA Specific Example

Given

A = 50
σ_A = 7.

This means the range of the number, or the uncertainty bars on a graph, are from min=50-σ=43 to max=50+σ=57. If one is plotting the natural log of 50 then the uncertainties bars are obtained from the natural log of the max and min values:

f	$=\ln(50)$	$= 3.912$
f_{min}	$=\ln(50-7)$	$= 3.761$
f_{max}	$=\ln(50+7)$	$= 4.043$

Note that the average difference between the min and the value and max and the value is:
3.912-3.761=0.151
4.043-3.912=0.131
uncertainty =average of the differences = (0.151+0.131)/2 = 0.141

B.3 General Error Propagation Using Differential Calculus

We will base our general error propagation on the Taylor expansion for a function:

B.3.1 $$f(x) = f(x_o) + (x - x_o)\frac{df(x)}{dx} + \frac{(x-x_o)^2}{2!}\frac{d^2 f(x)}{dx^2} + \ldots$$

This remarkable series expansion says that if you know the functional form, and the value of a function at some point, $x=x_o$, then you can find the value of the function anywhere by taking derivative of the function evaluated at $x=x_o$.

> (An easy example is the function $y=mx+b$. Here the only non-zero terms in the expansion are the first two, and it's easy to see that if you know the slope df/dx and intercept, $f(x_o)$, then you can find the value of the function anywhere.)

If we bring $f(x_o)$ to the left side of equation B.3.1 and define $\Delta f = f(x) - f(x_o)$ we can represent how the function changes as shown in figure B.3.1 below. The goal is to know $f(x)$, which is the value of the function at a point x which is near $f(x_o)$, our original point.

The Taylor expansion is saying:

- A first estimate of $f(x)$ is $f(x_o)$.
- A better estimate of $f(x)$ would include knowledge of the first derivative as is shown in the figure.
- Better estimates will be made by including higher derivatives.

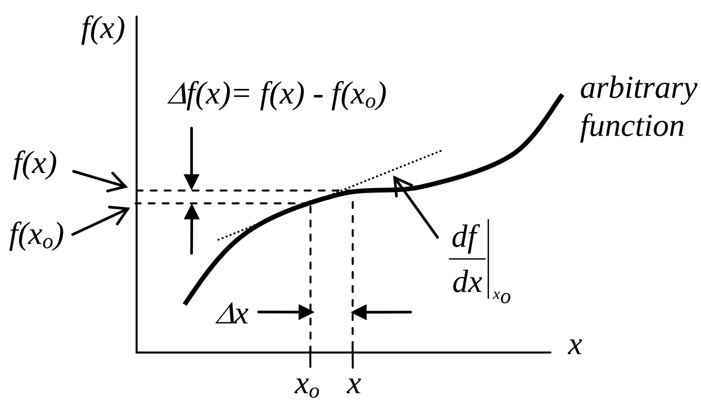


Figure B.3.1. Graphical Representation of Taylor Expansion.

For functions of more than one variable equation B.3.1 becomes,

B.3.2 $$\begin{aligned} f(x,y,\ldots) &= f(x_o,y_o,\ldots) + (x-x_o)\frac{\partial f}{\partial x} + (y-y_o)\frac{\partial f}{\partial y} + \ldots \\ &+ \frac{(x-x_o)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{(y-y_o)^2}{2!}\frac{\partial^2 f}{\partial y^2} + \ldots \end{aligned}$$

We wish to use this to consider small deviations of a function from a value (x_o, y_o) as our uncertainty. We thus bring the first term on the right side of equation B.3.2 to the left side, and define.

B.3.3 $\Delta f = f(x,y,\ldots) - f(x_o,y_o,\ldots)$.

We also define

B.3.4 $\Delta x = (x - x_o)$ $\Delta x = (y - y_o)$

If we consider small deviations we can drop the higher order terms (2nd derivative and higher) and we are left with,

B.3.5 $$\Delta f = \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}$$

In our data, we are considering not just one value, *x*, but a number of values obtained from multiple measurements. (Likewise for other variables, y, z, …). Thus the deviations, represented by Δ above, become average deviations from multiple measurements. That is, we will consider them as standard deviations resulting from multiple measurements. Specifically,

B.3.6 $$\sigma_x^2 = \lim_{N \to \infty} \sum (x_i - \bar{x})^2$$
$$\sigma_y^2 = \lim_{N \to \infty} \sum (y_i - \bar{y})^2 ,$$
etc

where $\bar{x}$ and $\bar{y}$ are the average values and x_i and y_i are the individual measurements.

Let us call the deviations in *x* and *y*, σ_x and σ_y, respectively. We then consider the variations, or uncertainties, in *f* to be, σ_f, (i.e., we let $\sigma_f \equiv \Delta f$), and, just as with the standard deviation, we calculate the square root of the deviations squared,

B.3.6 $$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x} + \sigma_y \frac{\partial f}{\partial y}\right)^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + 2\sigma_{xy}^2 \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$$

The term σ_{xy}^2, represents the product of deviations for the two variables *x* and *y*, which, again, is evaluated for each measurement. Specifically,

B.3.7 $$\sigma_{xy}^2 = \lim_{N \to \infty} \sum (x_i - \bar{x})(y_i - \bar{y})$$

If the variations in *x* and *y* are unrelated, (one would also say the uncertainties are independent, or uncorrelated), then the sum should become small with enough readings. (That is, over many readings there will be as many positive terms in the sum as negative terms and the net result will zero.) This is the case we will consider. (Although there certainly are cases when correlated uncertainties are relevant.)

Thus we assume uncorrelated errors and drop the last term in equation B.3.6. The final result for uncorrelated error becomes,

B.3.8 $$\boxed{\sigma_f = \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}}$$

Thus our final value for the function *f(x,y,z,..)* is reported as

B.3.9 $$f \pm \sigma_f$$

or

B.3.10 $$\boxed{f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}}$$

Equation B.3.10 is the most general way to determine the uncertainties when doing error propagation for uncorrelated errors.

Equation B.3.10 reduces to simpler equations for some special and common situations (adding, subtracting, multiplying and dividing). This will be explored next. The primary usefulness of equation B.3.10 in this form is for other, more complication, functions, especially those containing multiple variables. Some examples will be explored.

B.4 Adding or Subtracting Numbers with Independent Uncertainties

Given f = A + B A +/- σ_A B +/- σ_B

We use equation B.3.10: $f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$

Where x=A and y=B.

We first calculate the partial derivatives

$$\frac{\partial f}{\partial A} = 1$$
$$\frac{\partial f}{\partial B} = 1$$

Substituting these into equation B.3.10, we have

$$f \pm \sigma_f = A + B \quad \pm\sqrt{(\sigma_A)^2 + (\sigma_B)^2}$$

Thus, when adding (or subtracting) two numbers, the uncertainties combine in quadrature. Since addition and subtraction are common functions we write the above simplifications of the general error propagations:

B.4.1 Addition Rule $f \pm \sigma_f = A + B \quad \pm\sqrt{(\sigma_A)^2 + (\sigma_B)^2}$

B.4.2 Subtraction Rule $f \pm \sigma_f = A - B \quad \pm\sqrt{(\sigma_A)^2 + (\sigma_B)^2}$

Numerical Example:

Measure a position C as the sum of two numbers, A and B, with uncertainties, σ_A and σ_B, respectively.

C= A + B
A= 100 +/- 5 cm
B= 50 +/- 8cm
Total uncertainty is $\sigma_C = \sqrt{\sigma_A^2 + \sigma_B^2} = \sqrt{(5cm)^2 + (8cm)^2} = 9.43cm$

Final answer is C = 150 +/-9.43 cm

B.5 Multiplying or Dividing Numbers with Independent Uncertainties

B.5.1 Multiplication

Given f = AB A +/- σ_A B +/- σ_B

We use equation B.3.10: $$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Where x=A and y=B.

We first calculate the partial derivatives

$$\frac{\partial f}{\partial A} = B$$
$$\frac{\partial f}{\partial B} = A$$

Substituting these into equation B.4.9, we have

$$f \pm \sigma_f = AB \pm \sqrt{(B\sigma_A)^2 + (A\sigma_B)^2}$$

We can factor out AB from the radical to obtain,

$$f \pm \sigma_f = AB \pm AB\sqrt{\left(\frac{B\sigma_A}{AB}\right)^2 + \left(\frac{A\sigma_B}{AB}\right)^2}$$

or

B.5.1.1 $$f \pm \sigma_f = AB \pm AB\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$$

Note that for multiplication the *relative* uncertainties, $\frac{\sigma_x}{X}$, combine in quadrature.

Since multiplication is a common function we write the above simplification of the general error propagations a rule:

B.5.1.2 Multiplication Rule $$f \pm \sigma_f = AB \pm AB\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$$

B.5.2 Division

Given $f = \frac{A}{B}$ f = A/B A +/- σA B +/- σB

We use equation B.3.10: $f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$

Where x=A and y=B.

We first calculate the partial derivatives

$$\frac{\partial f}{\partial A} = \frac{1}{B}$$
$$\frac{\partial f}{\partial B} = -\frac{A}{B^2}$$

Substituting these into equation B.4.9, we have

$$f \pm \sigma_f = \frac{A}{B} \pm \sqrt{\left(\sigma_A \frac{1}{B}\right)^2 + \left(\sigma_B\left(-\frac{A}{B^2}\right)\right)^2} = \frac{A}{B} \pm \sqrt{\left(\frac{\sigma_A}{B}\right)^2 + \left(\frac{\sigma_B A}{B^2}\right)^2}$$

We can factor out A/B from the radical to obtain,

B.5.2.1 $$f \pm \sigma_f = \frac{A}{B} \pm \frac{A}{B}\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$$

Note that for division, like multiplication, the *relative* uncertainties, $\frac{\sigma_x}{X}$, combine in quadrature.

Since division is a common function we write the above simplification of the general error propagations a rule:

B.5.2.1 Quotient Rule $$f \pm \sigma_f = \frac{A}{B} \pm \frac{A}{B}\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$$

Numerical Example:

Measure a position C as the quotient of numbers, A and B, with uncertainties, σ_A and σ_B, respectively.

C= A/B
A= 100+/- 5 cm
B= 50+/- 3 sec

Total relative uncertainty is $\sqrt{\left(\frac{5}{100}\right)^2 + \left(\frac{3}{50}\right)^2} = 0.058$

Final answer is

C = (100cm)/(50sec) +/-(100cm)/(50sec)(0.058)

= (2 +/-0.116) cm/sec

B.6 Functions of a Single Variable

B.6.1 General Case

We use equation B.3.10: $f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$

If the function depends only on one variable then we have,

$$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2}$$

$$f \pm \sigma_x \frac{\partial f}{\partial x}$$

Since the function is a function of only one variable we write,

B.6.1.1 $$f \pm \sigma_x \frac{df}{dx}$$

This is equivalent to simply using differential calculus to vary the function. Some examples will demonstrate.

B.6.2 Quantities Raised to a Power

Given $f = x^n$ x +/- σ_x

Applying equation B.6.1.1,

$$f \pm \sigma_x \frac{df}{dx}$$

we calculate

$$\frac{df}{dx} = nx^{n-1}$$

So that the function with its uncertainty is,

B.6.2.1 Power Rule $$f \pm \sigma_f = x^n \pm nx^{n-1}\sigma_x$$

(This is also a common case to encounter, so we label it as a rule.)

Example: $C=A^3$ $A = 100cm$ $\sigma_A = 5$ cm

$C= A^3$
A= 100+/- 5 cm

Following equation B.6.2.1, the uncertainty in C is,

$$
\begin{aligned}
C \pm \sigma_C &= A^3 \pm 3A^2\sigma_A \\
&= (100cm)^3 \pm 3(100cm)^2(5cm) \\
&= (100cm)^3 \pm 1.5x10^5 cm^3 \\
&= 10^6\ cm^3 \pm 1.5x10^5 cm^3 \\
&= (10^6 \pm 1.5x10^5)cm^3
\end{aligned}
$$

Note on Correlated Errors
We can also write the above function in the form:

$$C = A^3 = A\ A\ A$$

Now we will treat C as the product of three identical functions. Clearly, since they are the same function, they are perfectly correlated. With this in mind we recalculate the uncertainty by using the equation that takes the correlation into account, and then by ignoring the correlation.

Method 1: Correlated Errors

We start by using equation B.3.6 with three variables,

$$\sigma_f^{2^2} = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + 2\sigma_{xy}^2 \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + 2\sigma_{xz}^2 \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} z + 2\sigma_{yz}^2 \frac{\partial f}{\partial y}\frac{\partial f}{\partial z}$$

Since the three variables are the same (*f=AAA*) we can simplify as follows,

$$\frac{\partial f}{\partial x} = \frac{\partial (xyz)}{\partial z} = xy = A^2$$

The correlation is found in the last three terms and we use equation B.3.7,

$$\sigma_{xy}^2 = \lim_{N \to \infty} \sum (x_i - \bar{x})(y_i - \bar{y})$$

The deviations, $x_i - \bar{x}$, we write as σ_x. Again, the since the three functions are identical we can write,

$$\sigma_{\mathbf{xy}} = \sigma_{\mathbf{xz}} = \sigma_{\mathbf{yz}} = \sigma_{\mathbf{A}}$$

Making these substitutions we have,

$$\sigma_C^2 = 3(\sigma_A A^2)^2 + 3(2\sigma_A^2 A^2 A^2) = 9\sigma_A^2 A^4$$

or

$$\sigma_C = \sqrt{9\sigma_A^2 A^4} = 3A^2\sigma_A$$

This is identical to our previous result where we used equation B.6.2.1 for the uncertainty of a function of one variable.

<u>Method 2: Incorrect Assumption of Uncorrelated Error</u>

Here we simply use equation B.3.8 and treat each of the A's as independent (we ignore the correlation). Equation B.3.8 yields,

$$\sigma_C = \sqrt{\left(\sigma_A \frac{\partial C}{\partial A}\right)^2 + \left(\sigma_A \frac{\partial C}{\partial A}\right)^2 + \left(\sigma_A \frac{\partial C}{\partial A}\right)^2} = \sqrt{3\left(\sigma_A \frac{\partial C}{\partial A}\right)^2} = \sqrt{3(\sigma_A A^2)^2} = \sqrt{3}A^2\sigma_A$$

This result is smaller that that derived using correlations. Thus if we ignore the correlations we underestimate the uncertainties.

B.7 Examples

In the examples that follow we will assume that the variables are independent. Thus we will use the equations for uncorrelated uncertainties.

B.7.1 Single Variable Functions

B.7.1.1 Constant of Proportionality **f**= ax a=constant x+/-σ_x

We use equation B.3.10: $$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Since *f* depends on only on variable this becomes,

$$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2} = f \pm \sigma_x \frac{\partial f}{\partial x} = f \pm \sigma_x \frac{df}{dx}$$

Thus we have

$$df/dx = a$$

so that our result will be reported as,

$$ax \text{ +/- } a\sigma_x$$

Numerical Example
Find uncertainty in f for:
f = 5x
x = 3.0
σ_x = 0.1
Solution:
f = 5(3.0) = 15.0
σ_f = 5(0.1) = 0.5

f = 15.0 +/- 0.5

Example B.7.1.2: **f=ln*x*** **x +/-σ_x**

We use equation B.3.10: $f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$

Since *f* depends on only on variable this becomes,

$$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2} = f \pm \sigma_x \frac{\partial f}{\partial x} = f \pm \sigma_x \frac{df}{dx}$$

We have, $f = \ln x$, so that $\frac{df}{dx} = \frac{1}{x}$

In terms of uncertainty this becomes,

B.7.1.2.1 $\ln x \pm \frac{\sigma_x}{x}$

Numerical Example

Measure a quantity C as the value of lnA, where A has uncertainty σ_A.

C= lnA
A= 100+/- 50 cm

Following equation B.7.1.2.1, the uncertainty in C is,

$$C = \ln A \pm \frac{\sigma_A}{A} = \ln(100) \pm \frac{50}{100} = 4.605 \pm 0.5$$

NOTE: We compare this to the max/min method:

f	=ln(100)	= 4.605
f_{min}	=ln(100-50)	= 3.912
f_{max}	=ln(100+5)	= 5.011

Note that the average difference between the min and the value and max and the value is:

4.605-3.912 = 0.6930
5.011-4.605 = 0.4056

uncertainty =average of the differences = (0.6930+0.4605)/2 = 0.577

Compare this to the calculus value of uncertainty found above, 0.50. We see that the max/min method yields a comparable result, but tends to overestimate the uncertainty.

B.7.2 Trigonometric Functions of One Variable

We use equation B.3.10: $$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Since *f* depends on only on variable this becomes,

$$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2} = f \pm \sigma_x \frac{\partial f}{\partial x} = f \pm \sigma_x \frac{df}{dx}$$

B.7.2.1 f = sinx

df/dx = cosx

so that

σ_f = cosx σ_x

Numerical Example x = 30^o +/- 2^o

Note that we must use radians for the uncertainty, so using 30^o = 0.52 radians, and 2^o = 0.035 radians, we write our values as

x = 0.52 σ_x = 0.035

so that

σ_x = cos(30^o)(0.035) = (0.866)(0.035) = 0.030

B.7.2.2 f = cosx

df/dx = -sinx

so that

$\sigma_f = -\sin x\, \sigma_x$

We see from equation B.3.10 above that the uncertainty gets squared before taking the square root. Thus, in this case we really want the absolute value,

$\sigma_f = \sin x\, \sigma_x$

B.7.2.3 f = tanx

df/dx = sec^{2}x

or

σ_f = sec^{2}x σ_x

B.7.2.4 f = x²cosx

We use equation B.3.10: $$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Since *f* depends on only on variable this becomes,

$$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2} = f \pm \sigma_x \frac{\partial f}{\partial x} = f \pm \sigma_x \frac{df}{dx}$$

$$\frac{df}{dA} = 2x\cos x - x^2 \sin x$$

So that our result is,

$$f \pm \sigma_f = f \pm \sigma_x \left(2x\cos x - x^2 \sin x\right)$$

B.7.3 Compound Functions

B.7.3.1 $f = Ae^{cB}$ with $A \pm \sigma_A$ $B \pm \sigma_B$ c=constant

We use equation B.3.10:

$$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Evaluating the partial derivatives,

$$\frac{\partial f}{\partial A} = e^{cB}$$
$$\frac{\partial f}{\partial B} = Ace^{cB}$$

Substituting these into equation B.3.10, we have

$$f \pm \sigma_f = Ae^{cB} \pm \sqrt{\left(e^{cB}\sigma_A\right)^2 + \left(Ace^{cB}\sigma_B\right)^2}$$

This result is sufficient for determining the uncertainty. We can simplify it if we like. We factor out Ae^{cB} from the radical,

$$f \pm \sigma_f = Ae^{cB} \pm Ae^{cB}\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(c\sigma_B\right)^2}$$

Note that if A were a constant then the function would be a function of one variable only. Thus, $\frac{\partial f}{\partial A} = 0$, and the uncertainty would reduce to,

$$f \pm \sigma_f = Ae^{cB} \pm Ace^{cB}\sigma_B$$

This is just what one would obtain as described in section B.6, by taking the differential of $f = Ae^{cB}$.

B.7.3.2 $f = x + bt^2$

$b = 7$

$x = 20.0$

$t = 4.0$

$\sigma_x = 0.3$

$\sigma_t = 0.1$

We have $f = 20.0 + 7(4.0)^2 = 132.0$

We use equation B.3.10: $$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Evaluating the partial derivatives,

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = 1$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} = 2\mathbf{bt}$$

Thus we have

$$\mathbf{f} \pm \sqrt{\left(\sigma_{\mathbf{x}} \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^2 + \left(\sigma_{\mathbf{t}} \frac{\partial \mathbf{f}}{\partial \mathbf{t}}\right)^2} = \mathbf{f} \pm \sqrt{\sigma_{\mathbf{x}}{}^2 + (\sigma_{\mathbf{y}} 2\mathbf{bt})^2}$$

$$\mathbf{f} \pm \sqrt{0.3^2 + ((0.1)(2)(7)(4.0))^2} = \mathbf{f} \pm \sqrt{0.3^2 + 5.6^2} = \mathbf{f} \pm 5.6$$

Thus the final result is stated as

f +/-σ_f = 132.0 +/-5.6

B.7.3.3 $\mathbf{f = x^2\cos\theta}$

We use equation B.3.10: $$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Evaluating the partial derivatives,

$$\frac{\partial f}{\partial x} = 2x\cos\theta$$

$$\frac{\partial f}{\partial \theta} = -x^2 \sin\theta$$

Substituting these into equation B.3.10, we have,

$$f \pm \sigma_f = x^2 \cos\theta \pm \sqrt{(\sigma_x 2x \cos\theta)^2 + (\sigma_\theta x^2 \sin\theta)^2}$$

(Here the minus sign in the second term is gone because that term gets squared.)

B.7.3.4 f = bxt²   b=constant   x +/-σ_x t +/-σ_t

We use equation B.3.10: $$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

We then have

$$\frac{\partial f}{\partial b} = 0$$
$$\frac{\partial f}{\partial x} = bt^2$$
$$\frac{\partial f}{\partial t} = 2bxt$$

Substituting these into equation B.3.10, we have

$$f \pm \sigma_f = bxt^2 \pm \sqrt{(\sigma_b \cdot 0)^2 + (\sigma_x bt^2)^2 + (\sigma_t 2bxt)^2}$$
$$f \pm \sigma_f = bxt^2 \pm \sqrt{(\sigma_x bt^2)^2 + (\sigma_t 2bxt)^2}$$

To simplify the form of this equation we factor out bxt^2 from the radical,

Note that this is just what we would have gotten if we had directly applied the product rule, equation B.5.1.1

$$f \pm \sigma_f = AB \pm AB\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$$

In this case we could have defined

$A = bx$
$B = t^2$

from which one obtains, using previously found rules for single variable uncertainties, (using differential calculus),

$\sigma_A = b\sigma_x$
$\sigma_B = 2t\sigma_t$

Substituting these into the product rule equation, B.5.1.1 we have

$$f \pm \sigma_f = bxt^2 \pm bxt^2 \sqrt{\left(\frac{b\sigma_x}{bx}\right)^2 + \left(\frac{2t\sigma_t}{t^2}\right)^2}$$

which easily reduces to our previous result,

$$f \pm \sigma_f = bxt^2 \pm bxt^2 \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\sigma_t \frac{2}{t}\right)^2}$$

Numerical example:

$f = bxt^2$
$b = 5$
$x = 2.0$
$t = 4.0$
$\sigma_x = 0.1$
$\sigma_t = 0.3$

$f = 5(2.0)(4.0)^2 = 160.0$

$$\sigma_f = 160.0\sqrt{\left(\frac{0.1}{2.0}\right)^2 + \left(\frac{2(0.3)}{4.0}\right)^2} = 160.0\sqrt{0.05^2 + 0.15^2} = 160.0(0.158) = 25.3$$

So that our final answer is

$f = 160.0$ +/- 25.3

B.7.3.5 f = zlnx z +/- $\boldsymbol{\sigma_z}$ x +/-$\boldsymbol{\sigma_x}$

We use equation B.3.10: $f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$

We then have

$$\frac{\partial f}{\partial z} = \ln x$$
$$\frac{\partial f}{\partial x} = z\left(\frac{1}{x}\right) = \frac{z}{x}$$

So that,

$$f \pm \sqrt{\left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \left(\sigma_x \frac{\partial f}{\partial x}\right)^2} = f \pm \sqrt{(\sigma_z \ln x)^2 + \left(\sigma_x \frac{z}{x}\right)^2}$$

Numerical Example

$z = 2$
$x = 3$
$\sigma_z = 0.1$
$\sigma_x = 0.2$

f = zlnx = 2ln(x) = 2.20

So that our result is,

$$2.20 \pm \sqrt{((0.1)(\ln 3))^2 + \left((0.2)\left(\frac{2}{3}\right)\right)^2} = 2.20 \text{ +/- } 0.173$$

B.7.3.6 f = lng g = x – 2z z +/- σ_z x +/-σ_x

We use equation B.3.10: $f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$

We thus have

$$f \pm \sqrt{\left(\sigma_g \frac{\partial f}{\partial g}\right)^2} = f \pm \sqrt{\left(\sigma_g \frac{1}{g}\right)^2}$$

We obtain σ_g by again using B.3.10,

$$\sigma_g = \sqrt{\left(\sigma_x \frac{\partial g}{\partial x}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2} = \sqrt{(\sigma_x (1))^2 + (\sigma_z (2))^2}$$

Numerical Example

$x = 95$
$z = 47$
$\sigma_x = 10$
$\sigma_z = 5$

We have

g = x – 2z = 95 – 2(47) = 1

$$\sigma_g = \sqrt{(\sigma_x (1))^2 + (\sigma_z (2))^2} = \sqrt{((10)(1))^2 + ((5)(2))^2} = \sqrt{100 + 100} = 14.1$$

Our final result is then

$$f \pm \sqrt{\left(\sigma_g \frac{1}{g}\right)^2} = f \pm \sqrt{\left(\sigma_g \frac{1}{g}\right)^2}$$

$$= \ln g \pm \sqrt{\left(\sigma_g \frac{1}{g}\right)^2}$$

$$= \ln(1) \pm \sqrt{\left((14.1)\frac{1}{1}\right)^2} = 0 \pm 14.1$$

We could also use *x* and *z* directly in the function and in B.3.10 as shown here.

We write f = ln(x – 2z)

Using B.3.10 we have $f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$

$$\frac{\partial f}{\partial x} = \frac{1}{x - 2z}$$
$$\frac{\partial f}{\partial z} = \frac{-1}{x - 2z}$$

So that,

$$f \pm \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2} = \ln(x - 2z) \pm \sqrt{\left(\sigma_x \frac{1}{x - 2z}\right)^2 + \left(\sigma_z \frac{-2}{x - 2z}\right)^2}$$

$$= \ln\left(95 - 2(47)\right) \pm \sqrt{\left((10)\frac{1}{95 - 2(47)}\right)^2 + \left(5\frac{-2}{95 - 2(47)}\right)^2}$$

$$= 0 \pm \sqrt{\left((10)\frac{1}{1}\right)^2 + \left(5\frac{-2}{1}\right)^2} = 0 \pm \sqrt{200} = 0 \pm 14.1$$

Which, of course, is the same answer as above.

B.8 Comparing Results

Typically, results of an experiment should be compared to accepted values, theoretical predictions or the results of others. One can simply calculate a percent difference between two values. One should also, see if the values are consistent with each other when the uncertainties are taken into account. **Both methods** should be reported if possible. Here are a few guidelines for reporting such comparisons.

B.8.1 Percent Difference

One way to compare results to a value obtained by someone else, including a value that is generally accepted as the best value of the scientific community, is to calculate the *percent difference* between the two. It is defined here.

B.8.1 $$percent \quad difference = \frac{|accepted \quad value - your \quad value|}{accepted \quad value} x100$$

Note: It is *not* generally good practice to refer to this as percent error. The reason for this is that the accepted value, or the value to which one is comparing, will also have some uncertainty. Calling the comparison simply the percent difference is the most straight forward statement to make without making any assumptions about a "true" value.

B.8.2 Overlap of Results

One should also compare results by noting the extent to which the results agree within the limits of the uncertainties in the results. If the uncertainties of the results are know then this comparison should be made.

B.8.3 Example

For example, suppose one measures the speed of light in two different experiments and compares it to the accepted value.

accepted value = $2.998x10^8$ m/s

measured value 1 = ($3.01x10^8$ +/- $0.1x10^8$)m/s

Percent difference: $$\frac{|2.998x10^8 - 3.01x10^8|}{2.998x10^8} x100 = 0.4\%$$

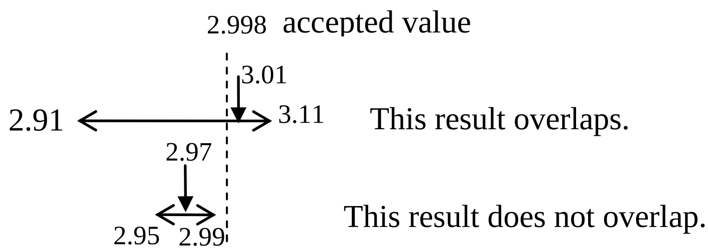
Overlap:	The result could be as high as	($3.01x10^8$ + $0.1x10^8$)m/s =	$3.01x10^8$ m/s
	or as low as	($3.01x10^8$ - $0.1x10^8$)m/s =	$2.91x10^8$ m/s

These values overlap with the accepted value;

measured value 2 = ($2.97x10^8$ +/- $0.02x10^8$)m/s

This could be as high as	($2.97x10^8$ + $0.02x10^8$)m/s =	$2.99x10^8$ m/s
or as low as	($2.97x10^8$ - $0.02x10^8$)m/s =	$2.95x10^8$ m/s

These values do not overlap with the accepted value (even though it is a more precise result). The figure below is a graphical representation of the overlapping.



<u>Correct ways to reports these results:</u>

value 1

Our result agrees with (or is consistent with) the accepted value within the limits of our uncertainties.

value 2

Our result does not agree with (or is inconsistent with) the accepted value within the limits of our uncertainties.

<u>Incorrect ways to reports these results:</u>

value 1

Our result is correct because it agrees with the accepted value within the limits of our uncertainties.

(To be scientifically precise, it is possible that the accepted value is "wrong", therefore one must be careful not to claim absolute knowledge.)

value 2

Our results agree with the accepted value because they are very close to the accepted value.

Unless one supports such a statement with further statistical analysis (such as using "confidence levels") it should not be made. If your results do not overlap within the limits uncertainties, then, although you can point out that the results are close, and calculate a percent difference, one should not claim that the results agree with the accepted value. Nor do they disagree. They simply fall outside the experimental uncertainty by a relatively small amount.)

A more thorough and precise comparison between results using uncertainties can be done using the idea of "confidence level", which will not be discussed here.

B.9 Summary

Table B.9.1 Table Summarizing Some Commonly Used Relationships

Quantity	**Equation**	**Section**
Average	$\bar{x} \equiv \frac{\sum_1^N x_i}{N}$	A.1.1
Standard Deviation	$\sigma \equiv \sqrt{\frac{\sum_1^N (x_i - \bar{x})^2}{N-1}}$	A.1.2
Adding Uncertainties in Quadrature	$\sigma_{tot} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + ...}$	A.3
Differential Changes	$df = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3 + \frac{\partial f}{\partial q_4} dq_4 + ...$	A.4
Error Propagation		
	All of the equations below come from one equation $\sigma_f = \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + ...}$	B.3
Adding *f* = *A* + *B* or Subtracting *f* = *A* - *B*	$f \pm \sigma_f = A + B \quad \pm \sqrt{(\sigma_A)^2 + (\sigma_B)^2}$ $f \pm \sigma_f = A - B \quad \pm \sqrt{(\sigma_A)^2 + (\sigma_B)^2}$	B.4
Multiplying *f* = *AB* or Dividing *f* = *A/B*	$f \pm \sigma_f = AB \pm AB\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$ $f \pm \sigma_f = \frac{A}{B} \pm \frac{A}{B}\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$	B.5.1 B.5.2
Quantities Raised to a Power $f = x^n$	$f \pm \sigma_f = x^n \pm n x^{n-1} \sigma_x$	B.6.2

B.10 Example of Error Propagation in an Experiment

Two carts of equal mass and equal but opposite velocity collide head on. Following the collision they stick together (perfectly inelastic collision) and are motionless.

We wish to experimentally test the theory of conservation of momentum that says,

$\mathbf{p}_1 + \mathbf{p}_2 = 0$

Motion is along a track, so that we need only consider one component of **p**.
We have p=mv

MEASUREMENTS: $m_1 = 0.505\text{kg}$ $m_2 = 0.510\text{kg}$
$v_1 = +0.65$ m/s $v_1 = -0.58$ m/s

Total momentum: $p_1 + p_2 = (0.505\text{kg})(+0.65\text{m/s}) + (0.510\text{kg})(-0.58\text{m/s})$
$= +0.328$ kgm/s -0.296 kgm/s
$= +0.0325$ kgm/s

UNCERTAINTIES:

mass $\sigma_{m1} = 0.05\text{kg}$ $\sigma_{m2} = 0.05\text{kg}$ Due to least count of mass scale.

velocity We do not measure velocity directly. We obtain it from $v = dx/dt$
We need to use our measured uncertainties in position and time to get our uncertainties in velocity.

Uncertainty in position: $\sigma_x = 0.01\text{m}$ Due to resolution of video image
Uncertainty in time: $\sigma_t = 0.033\text{s}$ Due to sampling rate of video camera

We use the equation

$$\sigma_f = \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Here we have $f = v = dx/dt$
$x = x$
$y = t$ so that

$$\sigma_v = \sqrt{\left(\sigma_x \frac{\partial\left(\frac{x}{t}\right)}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial\left(\frac{x}{t}\right)}{\partial t}\right)^2} = \sqrt{\left(\sigma_x \frac{1}{t}\right)^2 + \left(\sigma_t \frac{x}{t^2}\right)^2}$$

Since the carts are assumed to be moving with constant speeds we will use the total distance and time before the collision for one cart as our *x* and *t* values, and use the result as being the uncertainty for the speed of both carts. In general one might have to calculate the uncertainty for each measurement individually.

$x = 0.3m$, $t = 2.1s$, so that

$$\sigma_v = \sqrt{\left((0.01m)\frac{1}{(2.1s)}\right)^2 + \left((0.033s)\frac{(0.3m)}{(0.21s)}\right)^2} = 0.0053 \text{ m/s}$$

momentum $p = mv$

We use the equation

$$\sigma_f = \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Here we have $f = p = mv$
$x = m$
$y = v$ so that

$$\sigma_p = \sqrt{\left(\sigma_m \frac{\partial(mv)}{\partial m}\right)^2 + \left(\sigma_v \frac{\partial(mv)}{\partial v}\right)^2} = \sqrt{(\sigma_m v)^2 + (\sigma_v m)^2}$$

We apply this to each cart

$$\sigma_{p1} = \sqrt{(\sigma_{m1} v_1)^2 + (\sigma_v m_1)^2}$$
$$= \sqrt{((0.05kg)(0.65m/s))^2 + ((0.0053m/s)(0.505kg))^2}$$
$$= 0.049kgm/s$$

$$\sigma_{p2} = \sqrt{(\sigma_{m2} v_2)^2 + (\sigma_v m_2)^2}$$
$$= \sqrt{((0.05kg)(0.58m/s))^2 + ((0.0053m/s)(0.510kg))^2}$$
$$= 0.047kgm/s$$

total momentum $p = p_1 + p_2$

We use the equation

$$\sigma_f = \sqrt{\left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2 + \ldots}$$

Here we have $f = p = p_1 + p_2$
$x = p_1$
$y = p_2$ so that

$$\sigma_p = \sqrt{\left(\sigma_{p1}\frac{\partial(p_1+p_2)}{\partial p_1}\right)^2 + \left(\sigma_{p2}\frac{\partial(p_1+p_2)}{\partial p_2}\right)^2} = \sqrt{\sigma_{p1}{}^2 + \sigma_{p2}{}^2}$$

$$= \sqrt{0.049^2 + 0.047^2}\,kgm/s = 0.068kgm/s$$

FINAL RESULT:

$$p \pm \sigma_p = (0.0322 \pm 0.068)kgm/s$$

This result overlaps with the predicted value. (Which is zero in this case.) Our results are consistent with theory.

C.1 Tables

- **Number Table** Each table is numbered. The first table is Table 1.
- **Table Title** Each table has a title, which is unique among tables in the report.
- **Table Location** All tables are together, in order, after the text of the report.
- **Example Calculation** Include one example calculation for each type of calculation done. This can be done below the table or on a separate page labeled, “Example Calculations”.
- **Table Units** Table entries have correct units. Usually done in column headers.
- **Uncertainties** Tables should include data uncertainties in of each measurement (usually done in column headers or separate columns).
- **Make your own tables.** Do not copy from others or rip out the tables from the lab book.

Example

Table 2. Measurements of momentum from 3rd trial.		
mass (kg) +/-0.5kg	**speed (m/s) +/- 0.5**	**momentum (kg m/s)**
1.0	5.0	5.0
2.0	2.5	5.0

Example Calculation:

p=mv

(1.0kg)(5.0m/s)=5.0 kg m/s

C.2 Figures

C.1.1 Figure Format

• **Figure Numbers**	Each figure is **numbered**. The first figure is Figure 1.
• **Figure Titles**	Each figure has a **title**, which is unique among figures in the report.
• **Figure Location**	All figures are together, in order, after the tables.
• **Axis Labels**	Figures must include axis labels stating what the axis variables are, along with the appropriate units.
• **Axis Scales**	Figures must include an appropriate scale. The full scale should encompass the data. The data should take cover most of the range of the plot. The subdivisions of the scales must be easily rounded divisions. For example, if an axis scale went from 0-10, this could be broken up into a number of subdivisions. Some acceptable subdivisions are (0.1, 0.2, or 1, 2, 2, 5, etc). Some awkward subdivisions would be (0.16, 0.33)
• **Hand Figures**	Figures that are drawn must be drawn **quality graph paper** unless otherwise stated.
• **Uncertainties**	Figures must have an indication of the uncertainty of each point on the graph. Do this by including horizontal and vertical lines on each point on a graph indicating the uncertainty in each axis value. If the lines are too small to be seen on the scale being used, then state this either on the graph or in the text of the report. For example, in a position (x-axis) vs. time(y axis) plot suppose the uncertainty in position is 0.2 cm and the uncertainty in time is 0.5 sec. One draws a point at the correct coordinates. A horizontal line of length 0.2 cm is then drawn through the point and a vertical line of length 0.5 sec is drawn through the point.
• **Slope**	When calculating the slope **do not** use data points. Use points that lie on a line drawn through the data points.
• **Calculations**	When writing calculations on graphs, be sure to include units.
• **Make your own figures.**	Do not copy from others.

C.3 Graphical Analysis

C.3.1 Hand Fit to Linear Data

One wishes to draw a straight line (using a straight edge) through the data points in a way that best represents the trend of the data. This is a somewhat subjective procedure, yet, given the brain's keen ability to detect patterns, it can yield very reasonable results.

Steps

1. Calculate Centroid

An objective starting point is to calculate the *centroid* of the plotted data. The centroid is the point that is simply the average of the abscissa and ordinate values. In an x-y coordinate system the **centroid is just (x_{avg}, y_{avg})**, where for n data points,

$$x_{avg} = (x_1 + x_2 + x_3 + \ldots x_n)/n$$

and

$$y_{avg} = (y_1 + y_2 + y_3 + \ldots y_n)/n$$

2. Draw Max/Min Lines

Next, one draws two lines through the centroid. One line is drawn with the largest slope that reasonably follows the trend and this is subjective. The second line is drawn with the smallest slope that reasonably follows the trend. The best possible line will be lie in between these two extremes.

Note: One **does not have to encompass all data points** in the max/min lines. If this is done, the fluctuations, particularly near the centroid can cause one to draw lines that are too extreme and thus misrepresent the data.

3. Calculate Slopes of Max/Min Lines

From these two extreme lines one then calculates their slopes, which we will call, m_{low} and m_{high}. The slopes are calculated by taking any two locations on the lines that are drawn.

4. Calculate the "Best Slope"

The "**best" slope** is then found by taking an average:

$$m_{best} = \frac{m_{\max} + m_{\min}}{2},$$

5. Calculate the Uncertainty in the Best Slope

Calculate the absolute value of the difference between these slopes divided by 2 - this is the **uncertainty in the slope**:

$$\sigma_m = \left| \frac{m_{\max} - m_{\min}}{2} \right|.$$

Example

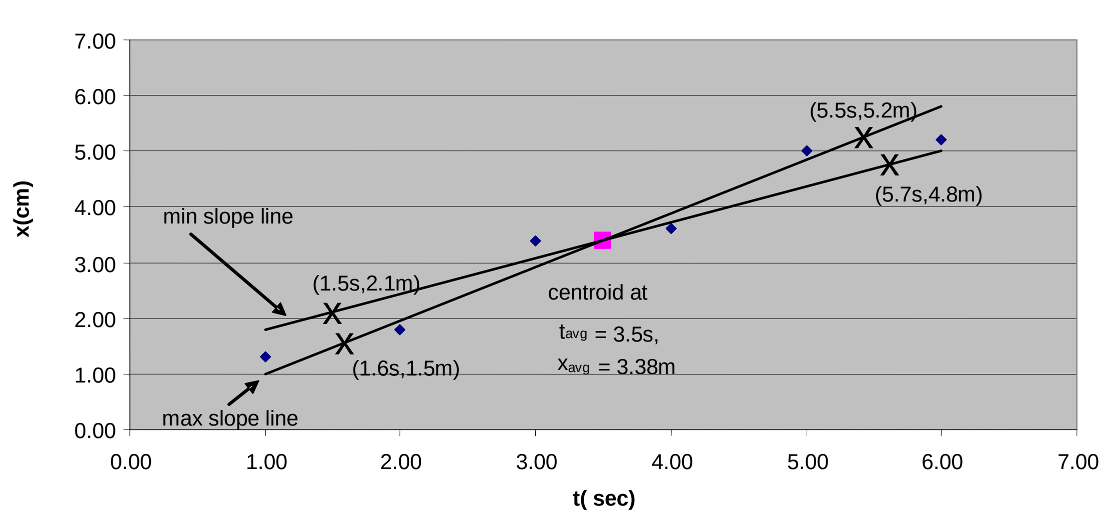
Consider the data set in table C.3.1.

Table C.3.1 Example Data Set for Fitting Line Slope by Hand

t(sec)	x(m)
1.00	1.30
2.00	1.80
3.00	3.40
4.00	3.60
5.00	5.00
6.00	5.20

The data points are plotted in figure C.3.1 as diamonds. Also shown in the figure are two lines representing the high and low sloped lines that would actually be drawn by hand with a straight edge. The slope of each line drawn is calculated using any two locations on the line. The locations used here are marked with "X"s. The high and low slope calculations are also shown as well as the best slope, m_{best}, and the uncertainty in slope, Δm.

Figure C.3.1 Example of Line Fit by Hand



$$t_{avg} = \frac{(1+2+3+4+5+6)s}{6} = 3.5s \qquad x_{avg} = \frac{(1.3+1.8+3.4+3.6+5+5.2)s}{6} = 3.38s$$

$$m_{\max} = \frac{(5.21-1.5)m}{(5.5-1.6)s} = 0.95m/s$$

$$m_{\min} = \frac{(4.8-2.1)m}{(5.7-1.5)s} = 0.64m/s$$

$$m_{best} = \frac{(m_{\max} + m_{\min})}{2} = 0.80m/s$$

$$\sigma_m = \frac{|m_{\max} - m_{\min}|}{2} = 0.155m/s$$

$$slope = 0.80 \pm 0.155m/s$$

C.3.2 Least Squares Unweighted Fit to Linear Data

If we are assuming the data follows a linear trend this means that the data has the form,

$$y = mx + b$$

where x is plotted on the abscissa (horizontal axis) and y on the ordinate (vertical axis). The slope is *m*, and the intercept is *b*. If all data points are equally weighted, that is, all data points are considered as having the same uncertainty, then, for n data points the slope and intercept can be found from the equations below.

$$y = mx + b$$

UNWEIGHTED

$$n = number \quad of \quad points$$

$$m = \frac{n\sum(x_i y_i) - \left(\sum x_i\right)\left(\sum y_i\right)}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$b = \frac{\left(\sum x_i^2\right)\left(\sum y_i\right) - \left(\sum x_i\right)\left(\sum x_i y_i\right)}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

C.3.3 Least Squares Weighted Fit to Linear Data

It is not uncommon for data points to have different uncertainties. In this case we want the data points with the least uncertainty to count, or weigh, the most when doing a fit to a linear trend. The equations below show how to find the slope an intercept for data that has uncertainties in *y*.

$$y = mx + b$$

$$WEIGHTED \quad FIT$$

$$\Delta y = error \quad in \quad y$$

$$W_i = \left(\frac{1}{\Delta y_i}\right)^2$$

$$m = \frac{\left(\sum W_i\right)\left(\sum W_i x_i y_i\right) - \left(\sum W_i x_i\right)\left(\sum W_i y_i\right)}{\left(\sum W_i\right)\left(\sum W_i x_i^2\right) - \left(\sum W_i x_i\right)^2} \qquad slope$$

$$b = \frac{\left(\sum W_i x_i^2\right)\left(\sum W_i y_i\right) - \left(\sum W_i x_i\right)\left(\sum W_i x_i y_i\right)}{\left(\sum W_i\right)\left(\sum W_i x_i^2\right) - \left(\sum W_i x_i\right)^2} \qquad \text{int}\, ercept$$

$$\Delta m = \sqrt{\frac{\sum W_i}{\left(\sum W_i\right)\left(\sum W_i x_i^2\right) - \left(\sum W_i x_i\right)^2}} \qquad uncerta\,\text{int}\, y \quad in \quad slope$$

$$\Delta b = \sqrt{\frac{\sum W_i x_i^2}{\left(\sum W_i\right)\left(\sum W_i x_i^2\right) - \left(\sum W_i x_i\right)^2}} \qquad uncerta\,\text{int}\, y \quad in \quad \text{int}\, ercept$$

$$yf_i = mx_i + b$$

$$\Delta yf_i = y_i - yf_i$$

$$\sigma y = \sqrt{\frac{\sum yf_i^{\,2}}{n} - \left(\frac{\sum yf_i}{n}\right)^2}$$

$$R = 1.0 - \frac{\sqrt{\frac{\sum \Delta yf_i^{\,2}}{n}}}{\sigma y}$$

D. Order-of-Magnitude Calculations

D.1 Introduction

One can often do calculations without the use of a computer or calculator by rounding and approximations, which produce an answer correct to within an order of magnitude. An order of magnitude is within a factor of 10. This is useful skill as it can save time and provide a quick and reasonable estimate before one proceeds to a more precise calculation.

Some Tools:

Rounding numbers
Binomial expansion

$$(1 \pm x)^{-n} = 1 \pm nx + \frac{n(n-1)}{2!}x^2 + \ldots \qquad |x| < 1$$

D.2 Examples

Given: $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ v=2.15x10^{7}m/s c=2.9997x10^{8}m/s

Approximations: v=2x10^{8}m/s c=3x10^{8}m/s Use binomial expansion on γ

$$\gamma \approx 1 - \left(-\frac{1}{2}\frac{v^2}{c^2}\right) = 1 + \frac{1}{2}\frac{(2\text{x}10^8\,\text{m/s})^2}{(3\text{x}10^8\,\text{m/s})^2} = 1 + \frac{1}{2}\frac{4}{9}\frac{10^{116}}{10^{16}} = 1 + \frac{2}{9} \approx 1.2$$

Calculator answer = 1.43
Answers differ by: 100x(1.2-1.43)/1.43 = 0.16%

Given: $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ v=1.59x10^{7}m/s c=2.9997x10^{8}m/s

Approximations: v=2x10^{7}m/s c=3x10^{8}m/s Use binomial expansion on γ

$$\gamma \approx 1 - \left(-\frac{1}{2}\frac{v^2}{c^2}\right) = 1 + \frac{1}{2}\frac{(2\text{x}10^7\,\text{m/s})^2}{(3\text{x}10^8\,\text{m/s})^2} = 1 + \frac{1}{2}\frac{4}{9}\frac{10^{14}}{10^{16}} = 1 + \frac{2}{9}x10^{-2} \approx 1 + 0.2\text{x}10^{-2} \approx 1.002$$

Calculator answer = 1.0014
Answers differ by: 100x(1.002-1.0014)/1.0014 = 0.06%

$\mathbf{F} = \frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$ q1=3.7x10^{-6}esu q1=15.0x10^{-6}esu r = 8x10^{-10}cm

$$F = \frac{(3.7x10^{-6}esu)(15x10^{-6}esu)}{(8x10^{-10})^2} \approx \frac{(4x10^{-6}esu)(15x10^{-6}esu)}{(8x10^{-10})^2} \approx \frac{(4)(15)(10^{-6})(10^{-6})}{(8)^2(10^{-10})^2}dynes$$

$$\approx \frac{(4)(15)(10^{-6})(10^{-6})}{64(10^{-20})}dynes \approx \frac{(60)(10^{-12})(10^{+20})}{64}dynes \approx 1X10^8 dynes$$

calculator answer: 8.67x10^{7}dynes
Answers differ by: 100x(1x10^{8}-8.67x10^{7}/8.67x10^{7} = 15%

D.3.Square Roots

Write the argument of the radical in scientific notation and then shift the decimal place so that the power of the exponent is even.

This leaves you with the square root of an even power of 10, which is easy, and a number between 0-100 where one can make a good estimate for the square root.

Calculate the square root of 427833

$$\sqrt{427833} = \sqrt{4.27833x10^5} \approx \sqrt{4.3x10^5} = \sqrt{43x10^4} = \sqrt{43x10^2x10^2} = \sqrt{43}\sqrt{10^2x10^2} = \sqrt{43}x10^2 \approx 6x10^2$$

calculator answer: 654
Answers differ by: 100x(600-654)/654 = 9%

Calculate the square root of 0.00137

$$\sqrt{0.00137} = \sqrt{1.37x10^{-3}} \approx \sqrt{1.4x10^{-3}} = \sqrt{14x10^{-2}} = \sqrt{14}\sqrt{10^{-2}} = \sqrt{14}x10^{-1} \approx 4x10^{-1}$$

calculator answer: 0.037
Answers differ by: 100x(0.04-0.037)/0.037 =8%

D.4 Exercises

1. Calculate $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ for v=10m/s
2. Calculate $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ for v=2.5x10^{8} m/s
3. Force between two charges where
   q1 = 2.78443x10^{-3}esu
   q2 = 5.21210098x10^{-4}esu
   r = 2.78443x10^{-8}cm
4. square root of 33378002
5. square root of 1/893

E. Exercises

E.1 Average and Standard Deviation

Consider the measurements of length shown in column 2 of Table E.1. Using the format of example A.1.2

Table E.1 Data for Finding Average and Standard Deviation

Data point	Value (sec)
1	55.6
2	54.1
3	54.8
4	55.2
5	56.3
6	55.2
7	53.1
8	54.8
9	54.7
10	55.3

E.2 Graphs of 1 Dimensional Kinematics

Consider the data in Table E.2.1 below.

Table E.2.1 Data for Position and Time

t (sec)	x (cm)
0.20	0.10
0.40	0.40
0.60	0.90
0.80	1.60
1.00	2.50
1.20	3.60
1.40	4.90
1.60	6.40
1.80	8.10
2.00	10.00
2.20	12.10
2.40	14.40
2.60	16.90
2.80	19.60
3.00	22.50
3.20	25.60
3.40	28.90
3.60	32.40
3.80	36.10

Make a graph, by hand, of position (ordinate) vs. time (abscissa).

Calculate the average velocity between times t=1.0 sec and t=3.0 sec.

Draw a smooth curve through your data points.
At time t=2.0 sec, draw a line tangent to the curve. Calculate the instantaneous velocity at this time by calculating the slope of the line. Put all calculations on the graph and, indicate which points on the line you have used in your calculation of slope.

Be sure to use the proper format for making a graph.

Be sure to include all units in your calculations.

E.3 Error Propagation Exercises

Refer to the example in Section B to help with these problems.

E.3.1 Find y and the uncertainty in y for the following function

$y = x_1 + x_2$

$x_1 = 5.0$

$x_2 = 10.0$

$\sigma_1 = 0.1$

$\sigma_2 = 0.2$

E.3.2 Find y and the uncertainty in y for the following function

$y = vt$

$v = 5.0$

$t = 10.0$

$\sigma_v = 0.1$

$\sigma_t = 0.2$

E.3.3 Find y and the uncertainty in y for the following function

$y = t^3$

$t = 10.0$

$\sigma_t = 0.2$

E.3.4 Find y and the uncertainty in y for the following function

$y = xt^3$

$x = 5.0$

$t = 10.0$

$\sigma_x = 0.1$

$\sigma_t = 0.2$

E.3.5 Find y and the uncertainty in y for the following equation for projectile motion

$y = y_o + v_{oy}t + ½at^2$

$y_o = 5.0$ m

$v_{oy}= 10.0$ m/s

$t = 3.0$ sec

$\sigma_{yo} = 0.1$ m

$\sigma_{voy} = 0.3$ m/s

$\sigma_t = 0.01$ sec

$a = 9.8$ m/s^2 with zero uncertainty

E.3.6 Find y and the uncertainty in y for the following function

$y = \cos\theta$

$\theta = 30^o$

$\sigma_\theta = 5^o$

E.3.7 Find y and the uncertainty in y for the following function

$y = x^2 \sin\theta$
$x = 5$ m
$\sigma_x = 2$ m
$\theta = 30^o$
$\sigma_\theta = 5^o$

E.3.8 Find y and the uncertainty in y for the following function

$y = \ln[\sin(x/z)]$
$x = 60$ m
$\sigma_x = 10$ m
$z = 2$ m
$\sigma_z = 0.5$m

E.4 Graphing Analysis Exercises

E.4.1 Fit by Hand

Plot the data in table E.4.1 by hand and fit the data by hand as outlined in section C.3.1

Table E.4.1 Data Set for Fitting Line Slope by Hand

t(sec)	x(m)
2.00	2.60
4.00	3.60
6.00	6.80
8.00	7.20
10.00	10.00

F. Trigonometric Identities

$$\sin u = \frac{1}{\csc u} \quad \cos u = \frac{1}{\sec u} \quad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \quad \sec u = \frac{1}{\cos u} \quad \cot u = \frac{1}{\tan u}$$

$$\sin^2 u + \cos^2 u = 1 \quad 1 + \tan^2 u = \sec^2 u \quad 1 + \cot^2 u = \csc^2 u$$

$$\tan u = \frac{\sin u}{\cos u} \quad \cot u = \frac{\cos u}{\sin u}$$

$$\sin(\frac{\pi}{2} - u) = \cos u \quad \cos(\frac{\pi}{2} - u) = \sin u \quad \tan(\frac{\pi}{2} - u) = \cot u$$

$$\csc(\frac{\pi}{2} - u) = \sec u \quad \sec(\frac{\pi}{2} - u) = \csc u \quad \cot(\frac{\pi}{2} - u) = \tan u$$

$$\sin(\frac{\pi}{2} - u) = \cos u \quad \cos(\frac{\pi}{2} - u) = \sin u \quad \tan(\frac{\pi}{2} - u) = \cot u$$

$$\csc(\frac{\pi}{2} - u) = \sec u \quad \sec(\frac{\pi}{2} - u) = \csc u \quad \cot(\frac{\pi}{2} - u) = \tan u$$

$$\sin(-u) = -\sin u \quad \cos(-u) = \cos u \quad \tan(-u) = -\tan u$$
$$\csc(-u) = -\csc u \quad \sec(-u) = \sec u \quad \cot(-u) = -\cot u$$

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Double Angle Formulas

$$\begin{aligned}\sin(2u) &= 2\sin u\cos u\\ \cos(2u) &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\\ \tan(2u) &= \frac{2\tan u}{1-\tan^2 u}\end{aligned}$$

Power-Reducing/Half Angle Formulas

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

$$\cos^2 u = \frac{1+\cos(2u)}{2}$$

$$\tan^2 u = \frac{1-\cos(2u)}{1+\cos(2u)}$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$

$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$

$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$

$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

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