Atkin's 8th Edition Self-test 9.7

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The following is Self-test 9.7 from Atikin's 8th Edition, page 295, done correctly. Thanks to Jared for catching my misapplication of the Hermite polynomial recursion relations.

1 What went wrong before?

Before we start, let's examine where things went wrong. The recursion relation for the Hermite polynomials is

$$yH_v = vH_{v-1} + \frac{1}{2}H_{v+1} \tag{1}$$

The is indubitably correct. However, when faced with a term like yH_{v-1} I incorrectly said

$$yH_{v-1} = vH_{v-2} + \frac{1}{2}H_v \tag{2}$$

which, as Jared pointed out, is not correct. The *correct* relationship in this case is

$$yH_{v-1} = (\mathbf{v-1})H_{v-2} + \frac{1}{2}H_v \tag{3}$$

Other recursion relations can be generated in the analogous way. With this knowledge in hand, let's proceed to calculate $\langle x^2 \rangle$ for the quantum harmonic oscillator.

2 $\left\langle x^2 \right\rangle$ for the quantum harmonic oscillator

$$\begin{aligned} \left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} N_v H_v\left(\frac{x}{\alpha}\right) \mathrm{e}^{-\frac{x^2}{2\alpha}} x^2 N_v H_v\left(\frac{x}{\alpha}\right) \mathrm{e}^{-\frac{x^2}{2\alpha}} \mathrm{d}x & (4) \\ &= N_v^2 \int_{-\infty}^{\infty} H_v\left(\frac{x}{\alpha}\right) x^2 H_v\left(\frac{x}{\alpha}\right) \mathrm{e}^{-\frac{x^2}{\alpha}} \mathrm{d}x & (5) \end{aligned}$$

Substitute

$$\begin{aligned} y &= \frac{x}{\alpha} & (6) \\ x &= \alpha y & (7) \\ \mathrm{d}x &= \alpha \mathrm{d}y & (8) \end{aligned}$$

$$\left\langle x^2 \right\rangle = N_v^2 \alpha^3 \int_{-\infty}^{\infty} H_v(y)\, y^2 H_v(y)\, \mathrm{e}^{-y^2} \mathrm{d}y \tag{9}$$

Now we use the recursion relations

$$
\begin{aligned}
y^2 H_v &= yvH_{v-1} + \frac{y}{2}H_{v+1} && (10)\\
yvH_{v-1} &= v(v-1)H_{v-2} + \frac{v}{2}H_v && (11)\\
\frac{y}{2}H_{v+1} &= \frac{v+1}{2}H_v + \frac{1}{4}H_{v+2} && (12)
\end{aligned}
$$

$$
\left\langle x^2 \right\rangle = N_v^2 \alpha^3 \int_{-\infty}^{\infty} H_v \left[v(v-1)H_{v-2} + \frac{v}{2}H_v + \frac{v+1}{2}H_v + \frac{1}{4}H_{v+2} \right] \mathrm{e}^{-y^2} \mathrm{d}y \qquad (13)
$$

Terms involving Hermite polynomials of different v will be zero due to orthogonality.

$$
\begin{aligned}
\left\langle x^2 \right\rangle &= N_v^2 \alpha^3 \int_{-\infty}^{\infty} H_v \left[\frac{v}{2}H_v + \frac{v+1}{2}H_v \right] \mathrm{e}^{-y^2} \mathrm{d}y && (14)\\
&= N_v^2 \alpha^3 \int_{-\infty}^{\infty} H_v \left[\left(v + \frac{1}{2} \right) H_v \right] \mathrm{e}^{-y^2} \mathrm{d}y \quad \boxed{\text{There's the } v + \tfrac{1}{2} \text{ we were looking for!}} && (15)\\
&= N_v^2 \alpha^3 \left(v + \frac{1}{2} \right) \int_{-\infty}^{\infty} H_v H_v \mathrm{e}^{-y^2} \mathrm{d}y && (16)\\
&= \frac{1}{\alpha \sqrt{\pi} 2^v v!} \alpha^3 \left(v + \frac{1}{2} \right) \sqrt{\pi} 2^v v! && (17)\\
&= \boxed{\alpha^2 \left(v + \frac{1}{2} \right)} && (18)
\end{aligned}
$$

Now that you know how to correctly apply the Hermite polynomial recursion relations multiple times, calculating $\langle x^3 \rangle$ and $\langle x^4 \rangle$ should be simple.