Iterative Regularization and its Applications

Muhammad Aslam

Department of Mathematics

Lock Haven University of Pennsylvania

Outline of the Talk

- Problem Setting
  Inverse Problem is introduced.
- Regularization Approach
  Solution setup via Regularization Methods.
- Iterative Regularization
  Why Iterative Regularization approaches are important?
- Existing Iterative Regularization Approaches
  Some existing Iterative Regularization approaches and some interesting questions will be discussed.
- Proposed Iterative Regularization Methods
  New proposed methods will be discussed.
- Applications: Image Denoising $\&$ Edge Detection
  Numerical results will be presented.

Part I:

Problem Setting

Inverse Problem

- Introduction

  Let X be a true image and V is some error (random noise, film granularity or some disturbance). Then the acquired image Y can be expressed as,

$$Y = X + V$$

Inverse Problem

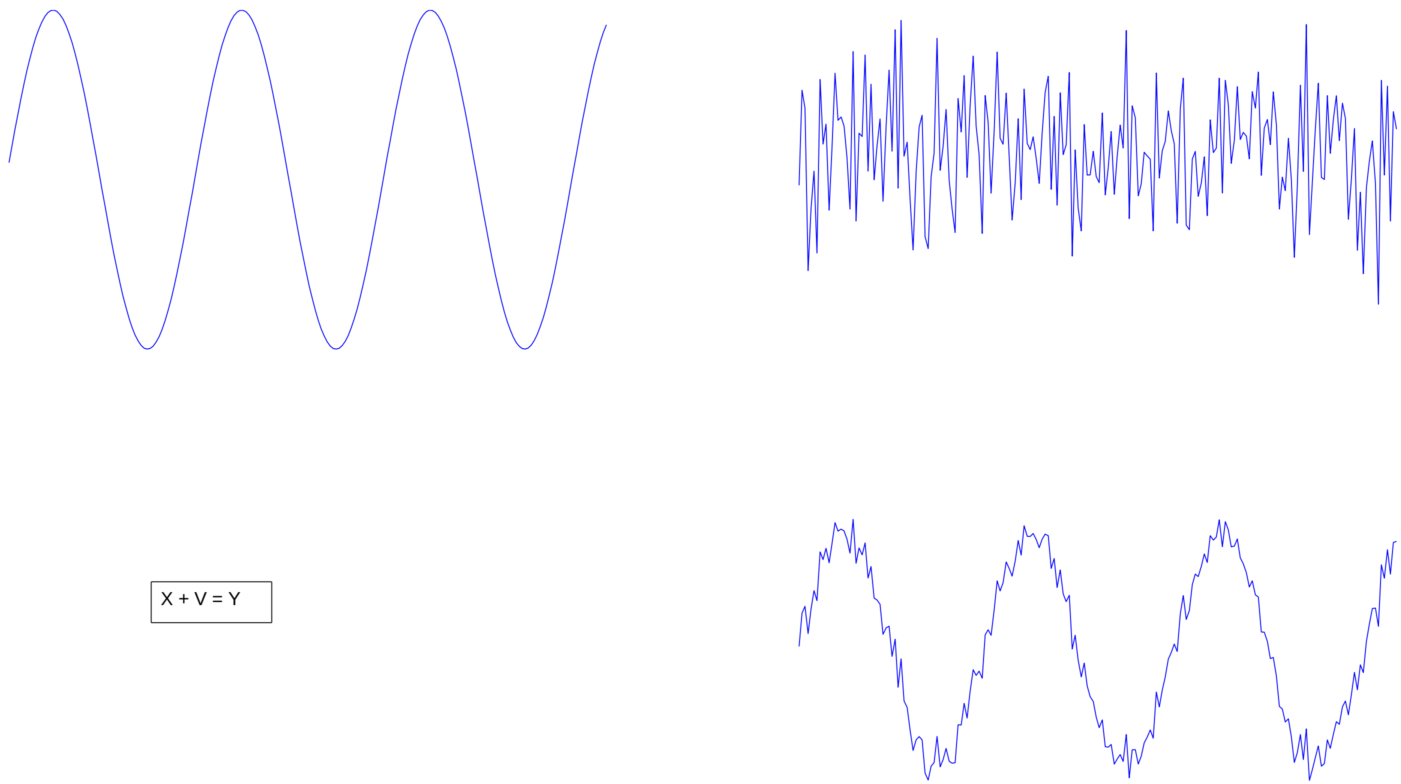
- Introduction

  Let X be a true image and V is some error (random noise, film granularity or some disturbance). Then the acquired image Y can be expressed as,

$$Y = X + V$$

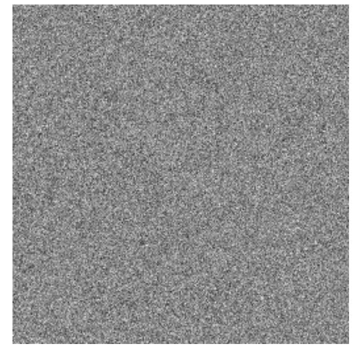
- Inverse Problem
  - Only the acquired image Y is the known information.
  - We have to determine the true image X.

Inverse Problem



Inverse Problem

+



=

Part II:
Regularization Approach

Regularization Approach

- Define a cost function

  $C(X,Y) = H(X,Y) + J(X)$

  where $H(X,Y) = \frac{1}{2}||X - Y||_2^2$ and $J(X)$ is a regularization functional or penalty term and it can have different forms.

Regularization Approach

- Define a cost function

  $C(X,Y) = H(X,Y) + J(X)$

  where $H(X,Y) = \frac{1}{2}||X - Y||_2^2$ and $J(X)$ is a regularization functional or penalty term and it can have different forms.

- Commonly used Regularization Functionals
  - *Tikhonov* $J(X) = \frac{\alpha}{2}||X||_2^2$
  - *Total Variation* $J(X) = \alpha||\nabla X||_1^2$

Regularization Approach

- Regularization Approach
  - By regularization approach, the estimate of X can be found by minimizing the above defined cost function $C(X,Y)$ for the given value of Y.
    $\widehat{X} = \arg\min_{X} C(X,Y) = \arg\min_{X}\{\frac{1}{2}||X - Y||_2^2 + J(X)\}$

Regularization Approach

- Regularization Approach
  - By regularization approach, the estimate of X can be found by minimizing the above defined cost function $C(X,Y)$ for the given value of Y.
    $\widehat{X} = \arg\min_{X} C(X,Y) = \arg\min_{X}\{\frac{1}{2}||X - Y||_2^2 + J(X)\}$
- Why Regularization Functional is important?
  - If we take $J(X) = 0$, then we have the trivial solution of $\widehat{X} = Y$.

Part III:
Iterative Regularization

Iterative Regularization

- Why Iterative Regularization Approach?
  - Iterative regularization methods improve the traditional regularization estimate.
  - It can be formulated by updating the cost function of choice iteratively.

Iterative Regularization

- Why Iterative Regularization Approach?
  - Iterative regularization methods improve the traditional regularization estimate.
  - It can be formulated by updating the cost function of choice iteratively.
- Mathematical Expression

$$X_{k+1} = \arg\min_{X} C_k(X, Y)$$

Iterative Regularization

- Why Iterative Regularization Approach?
  - Iterative regularization methods improve the traditional regularization estimate.
  - It can be formulated by updating the cost function of choice iteratively.
- Mathematical Expression

$$X_{k+1} = \arg\min_{X} C_k(X, Y)$$

- The estimate at each iteration can be thought of as the minimizer of a revised cost function $C_k(X, Y)$.

Part IV:
Existing Iterative Regularization Approaches

Existing Iterative Regularization

- Using the operator $\beta[\cdot]$ to denote the net effect of the minimization in $X_1 = \arg\min_X C(X, Y)$, that is $X_1 = \beta[Y]$ then the expressions for the most recent iterative regularization approaches are;

Existing Iterative Regularization

- Using the operator $\beta[\cdot]$ to denote the net effect of the minimization in $X_1 = \arg\min_X C(X,Y)$, that is $X_1 = \beta[Y]$ then the expressions for the most recent iterative regularization approaches are;
- Osher, Burger, Goldfrab, Xu, Yin

$$X_{k+1} = \beta[Y + \sum_{i=1}^{k}(Y - X_i)]$$

Existing Iterative Regularization

- Using the operator $\beta[\cdot]$ to denote the net effect of the minimization in $X_1 = \arg\min_X C(X,Y)$, that is $X_1 = \beta[Y]$ then the expressions for the most recent iterative regularization approaches are;
- Osher, Burger, Goldfrab, Xu, Yin

$$X_{k+1} = \beta[Y + \sum_{i=1}^{k}(Y - X_i)]$$

- Tukey

$$X_{k+1} = X_k + \beta[Y - X_k] = \beta[Y] + \sum_{i=1}^{k}\beta[Y - X_i]$$

Existing Iterative Regularization

- Charest, Elad, Milanfar Method I

$$X_{k+1} = \beta[Y] + \beta[\sum_{i=1}^{k}(Y - X_i)]$$

Existing Iterative Regularization

- Charest, Elad, Milanfar Method I

$$X_{k+1} = \beta[Y] + \beta[\sum_{i=1}^{k}(Y - X_i)]$$

- Charest, Elad, Milanfar Method II

$$X_{k+1} = X_k + \beta[Y] - \beta[X_k] = \beta[Y] + \sum_{i=1}^{k}(\beta[Y] - \beta[X_i])$$

Existing Iterative Regularization

- Two Important Questions

Existing Iterative Regularization

- Two Important Questions
- Since $\beta[\cdot]$ is nonlinear in general but if we consider it linear then all the methods are identical. *This is the unanswered question*.

Existing Iterative Regularization

- Two Important Questions
- Since $\beta[\cdot]$ is nonlinear in general but if we consider it linear then all the methods are identical. *This is the unanswered question*.
- All of the above methods weakly converge to Y.

Part V:
Proposed Iterative Regularization Methods

Proposed Iterative Regularization

- Proposed A

$$X_{k+1} = \beta[X_k] + \beta[\sum_{i=1}^{k}(Y - X_i)]$$

Proposed Iterative Regularization

- Proposed A

$$X_{k+1} = \beta[X_k] + \beta[\sum_{i=1}^{k}(Y - X_i)]$$

- Proposed B

$$X_{k+1} = \beta[X_k] + \sum_{i=1}^{k}\beta[Y - X_i]$$

Proposed Iterative Regularization

- Proposed A

$$X_{k+1} = \beta[X_k] + \beta[\sum_{i=1}^{k}(Y - X_i)]$$

- Proposed B

$$X_{k+1} = \beta[X_k] + \sum_{i=1}^{k}\beta[Y - X_i]$$

- Note: If we consider $\beta[\cdot]$ as a linear then above two methods are identical but different from the previous methods.

Part VI:

Numerical Results

Image Denoising

Noisy Image

First Iteration

2nd Iteration

3rd Iteration

Image Denoising

Noisy Image

5th Iteration

Noise removed in 5 iterations

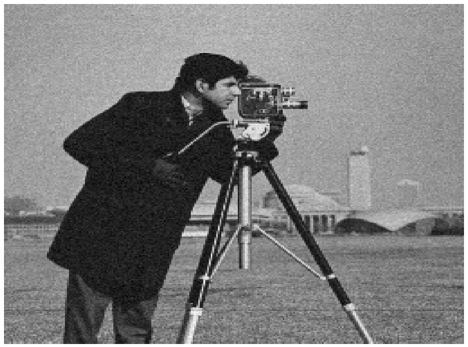
Image Denoising

Noisy Image

10th Iteration

Noise removed in 10 iterations

Edge Detection



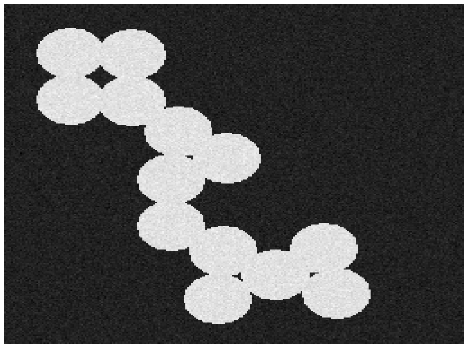
Noisy Image

After 3 iteration

Detected Edges

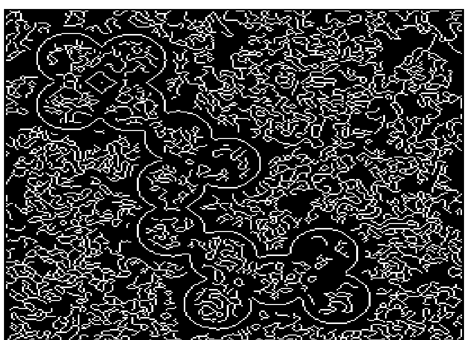
Detected Edges

Edge Detection

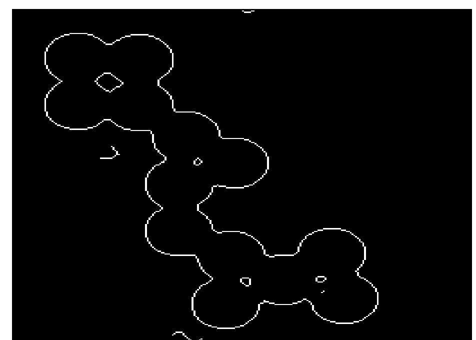


Noisy Image

Smoothed image
after 3 iterations



Detected Edges



Detected Edges

Thank You!