Math 141: Integration

Dr. Richard Mikula

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The Antiderivative of a function $f(x)$:

we say that $F(x)$ is an **antiderivative** for a function $y = f(x)$ if

$$F'(x) = f(x).$$

Clearly, if $F(x)$ is an antiderivative for $f(x)$, then so is $F(x) + C$, $C \in \mathbb{R}$. Thus antiderivatives are not unique. Moreover, all possible antiderivatives for a given function $f(x)$ are given by the set

$$\{F(x) + C : C \in \mathbb{R}\}$$

where $F(x)$ is any antiderivative for $f(x)$. The set of all antiderivatives for a function $f(x)$ is usually denoted by

$$\int f(x)dx$$

and is also referred to by the **indefinite integral** of $f(x)$.

We now make a list of some well-known indefinite integrals, where C here represents any real number:

$f(x)$	$\int f(x)dx$
0	C
1	$x + C$
$x^n, \quad n \neq -1$	$\frac{1}{n+1}x^{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

Some Examples: Find the indefinite integral for the following functions.

- $2x^{\frac{1}{3}} - 5x^{-\frac{1}{2}}$

- $4\sin x + 3\cos x$

Solution:

$$\int 2x^{\frac{1}{3}} - 5x^{-\frac{1}{2}} dx = 2\frac{x^{\frac{4}{3}}}{\frac{4}{3}} - 5\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$\int 4\sin x + 3\cos x dx = -4\cos x + 3\sin x + C.$$

Finding f if you know f':

The principle to keep in mind here is

$$\int f'(x)dx = f(x) + C.$$

If we are given some info about the value of f at a particular point in the domain, then you may say what f is exactly, when you also know $f'(x)$. Such problems are commonly referred to as **initial value problems**.

Example: Suppose

$$f'(x) = x^2 + 7$$

a: Find the possible form of $f(x)$. **b:** If $f(0) = 1$, find $f(x)$ explicitly.

Solution:

$$\int f'(x)dx = f(x) + C$$

and thus

$$f(x) + C = \int x^2 + 7dx$$

$$= \frac{x^3}{3} + 7x$$

and thus

$$f(x) = \frac{x^3}{3} + 7x + C$$

for some constant C. If in addition, we assume $f(0) = 1$, then we find out that

$$1 = f(0) = \frac{0^3}{3} + 7 \cdot 0 + C = C$$

and thus

$$f(x) = \frac{x^3}{3} + 7x + 1.$$

Applications to Physics:

In physics, if we have an object moving along a straight line, or moving under the influence of a force that is being applied in only one direction (along a straight line), we may use our knowledge of calculus to model the motion of the object.

Here, if $s(t)$ represents the position of the object at time t, then the velocity is given by $v(t) = s'(t)$, the acceleration is given by $a(t) = v'(t) = s''(t)$ and the speed of the object is $|v(t)|$.

Example: A ball is thrown up in the air with an initial speed of $3\frac{meters}{second}$, and initial position 1.5 meters off the ground. Find the function that describes the motion of the ball at any time t.

Solution: Here we shall assume that the only force acting on the ball is the force due to gravity. **Newton's Second Law of Motion** says the the sum of the forces acting on an object must equal $m \cdot a$, where a is the acceleration of the object and m is the mass of the object. Now, we must realize that the *force due to gravity* acting on an object is given by $-9.8m$ if we work in metric units and seconds and $-32m$ if we work in English units and seconds. Here the minus sign is toward the direction of the center of the earth.

Putting this together, we see that

$$ma = -9.8m$$

and thus

$$a = -9.8\frac{meters}{second^2}.$$

Now

$$v(t) = \int a(t)dt$$

$$= \int -9.8dt$$

$$= -9.8t + C_1.$$

Thus

$$s(t) = \int v(t)dt$$

$$= \int -9.8t + C_1 dt$$

$$= -4.9t^2 + C_1 t + C_2.$$

Now we need to use our initial conditions:

We are told that the initial speed is 3, and the ball is being thrown up in the air. Thus

$$v(0) = 3.$$

On the other hand, by our derived formula $v(t) = -9.8t + C_1$ we find that

$$3 = v(0) = -9.8 \cdot 0 + C_1 = C_1$$

and thus

$$s(t) = -4.9t^2 + 3t + C_2.$$

We now use our second initial condition, namely $s(0) = 1.5$ to get

$$1.5 = s(0)$$

$$= -4.9 \cdot 0^2 + 3 \cdot 0 + C_2$$

and thus

$$s(t) = -4.9t^2 + 3t + 1.5.$$

We are now in the position to answer some more questions:

- Find the maximum height the ball reaches, and the time it reaches this height.
- Find the time the ball hits the ground.

To answer the first question, we must realize the the ball hits its maximum height when $v = 0$. Thus, we need to solve

$$v(t) = -9.8t + 3 = 0$$

and thus $t = \frac{3}{9.8}$. Thus, the maximum height is

$$s(\frac{3}{9.8}) = -4.9 \cdot (\frac{3}{9.8})^2 + 3 \cdot \frac{3}{9.8} + 1.5$$

$$\approx 1.959 meters.$$

The time the ball hits the ground is given by a solution to $s(t) = 0$. Thus we need to solve

$$-4.9t^2 + 3t + 1.5 = 0$$

$$\Longleftrightarrow \quad t = \frac{3 \pm \sqrt{38.4}}{9.8}$$

or simply

$$t \approx 0.938 \quad or \quad t \approx -0.3262.$$

Clearly the first t value is our desired time.

Areas Under the Curve and the Definite Integral:

Here we are faced with the problem of trying to find areas of regions in the plane. We first consider regions which are bounded by the x-axis, the graph of a non-negative function $y = f(x)$, and two vertical lines $x = a$, $x = b$, for $a < b$ in the domain of our function.

To find such areas, we must draw a figure to see the key idea (this will be done in lecture, see p. 198 for instance in the text for the general idea).

What we due is break up the interval $[a, b]$ into n pieces (where $n \in \mathbb{N}$ will be some large integer) of equal length

$$\triangle x = \frac{b-a}{n}.$$

We define the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

as $x_1 = a + \triangle x$, $x_2 = x_1 + \triangle x$, $\cdots$, $x_i = x_{i-1} + \triangle x$, $\cdots$, $x_n = x_{n-1} + \triangle x = b$.

We look at the portion of the figure (whose area A we wish to compute) that comes from the area under the curve $y = f(x)$ over the interval $[x_{i-1}, x_i]$. * If n is quite large, and our function $f(x)$ is continuous, then the area of this piece may be approximated by the area of a rectangle, whose base has length $\triangle x$, and whose height is $f(x_i^*)$, where $x_i^* \in [x_{i-1}, x_i]$ is any point in this interval. Thus the area of this little sliver is roughly

$$f(x_i^*)\triangle x.$$

*This is a tiny sliver of the total piece whose area is A.

Doing this over each interval $[x_{i-1}, x_i]$ for $i = 1, 2, \cdots, n$, and adding up the areas of the approximating rectangles, we get

$$A \approx f(x_1^*)\triangle x + f(x_2^*)\triangle x + \cdots + f(x_n^*)\triangle x$$

$$= \sum_{i=1}^{n} f(x_i^*)\triangle x.$$

As $n \to \infty$, one can show that this sum converges to the area A *. This limit† is called the **definite integral** of $f(x)$ over the interval $[a, b]$ and is denoted by

$$\int_a^b f(x)dx.$$

*at least when f is continuous

†which is actually equal to A

Example: Let us do this for the region bounded by $y = x^2$, the x-axis and the lines $x = 0$ and $x = 1$. We first do this with $n = 5$ and x_i^* equal to x_i, and $\triangle x = \frac{1}{5}$. Thus the points $\{x_i\}$ are

$$a = x_0 = 0, x_1 = 0.2, x_2 = 0.4,$$

$$x_3 = 0.6, x_4 = 0.8, x_5 = 1.$$

Hence the area we wish to compute may be approximated by

$$0.2^2 \cdot 0.2 + 0.4^2 \cdot 0.2 + 0.6^2 \cdot 0.2 + 0.8^2 \cdot 0.2 + 1^2 \cdot 0.2$$

$$= 0.44.$$

Now, we do this for a general n, with the same x_i^*. Hence $\triangle x = \frac{1}{n}$ and thus $x_i = \frac{i}{n}$ for $i = 1, 2, \cdots, n$. Thus

$$A \approx (\frac{1}{n})^2\frac{1}{n} + (\frac{2}{n})^2\frac{1}{n} + \cdots + (\frac{i}{n})^2\frac{1}{n}$$

$$+ \cdots + (\frac{n}{n})^2\frac{1}{n}$$

$$= \frac{1}{n^3}\{1^2 + 2^2 \cdots + i^2 + \cdots + n^2\}$$

$$= \frac{1}{n^3}\{\frac{n(n+1)(2n+1)}{6}\}.^*$$

$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

Letting $n \to \infty$ we find that this converges to $\frac{1}{3}$, and this is the desired area.

*Here we used a formula in the text on page 196.

The quantity

$$\sum_{i=1}^{n} f(x_i^*)\triangle x$$

is referred to as a **Riemann Sum** for the function $y = f(x)$ on the interval $[a, b]$. In the previous discussion, we assumed the function was non-negative over this interval. Here, this Riemann Sum approximates the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$, $x = b$. If f is continuous, one can show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x = \int_a^b f(x)dx$$

exists, and equals the area of the region so described*.

*Here it is important to realize that our limit is taken over all possible Riemann sums, as $n \to \infty$.

We may actually extend the above definition of a definite integral for a function $y = f(x)$ on an interval $[a, b]$ to functions that are not-necessarily non-negative. It is just that this definite integral $\int_a^b f(x)dx$ will no longer represent an area.

Moreover, we can actually show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x = \int_a^b f(x)dx$$

exists whenever f is continuous, and even more generally if f is continuous except a finite number of points, where it has a jump discontinuity*.

*That is the function has left and right handed limits existing, but they are not the same value.

There are some important properties of the definite integral that we should note*:

- **Linearity**

$$\int_a^b \alpha f(x)+\beta g(x)dx = \alpha \int_a^b f(x)dx+\beta \int_a^b g(x)dx$$

-

$$\int_a^a f(x)dx = 0$$

-

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

-

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

*assuming all integrals listed do indeed exist

- $$f(x) \geq g(x)$$

  implies

  $$\int_a^b f(x)dx \geq \int_a^b g(x)dx,$$

  and thus

  $$m \leq f(x) \leq M$$

  implies

  $$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

The Fundamental Theorem of Calculus:

Suppose $f(x)$ is a continuous function on the interval $[a, b]$. Then, we know that the definite integral

$$\int_a^b f(x)dx$$

exists. Moreover, for any number $x \in [a, b]$ we actually know that $g(x)$ defined by the integral $\int_a^x f(t)dt$ is well-defined, since each of these integrals exists. We now note the following important theorem:

Theorem: For f continuous on $[a, b]$, the function

$$g(x) = \int_a^x f(t)dt$$

is differentiable for any $x \in (a, b)$, and

$$g'(x) = f(x).$$

Thus g is an antiderivative for f.

proof: We wish to show $\frac{g(x+h)-g(x)}{h}$ approaches $f(x)$ as $h \to 0$.

Or equivalently,

$$\frac{g(x \pm h) - g(x)}{\pm h} \to f(x) \quad as \quad h \to 0^+.$$

For $h > 0$ we have

$$g(x+h) - g(x) = \int_x^{x+h} f(t)dt,$$

and

$$g(x-h) - g(x) = -\int_{x-h}^{x} f(t)dt.$$

Now, let $\varepsilon > 0$ be given. By continuity of f at x, we know that there exists $\delta > 0$ so that

$$x - \delta < z < x + \delta \Rightarrow f(x) - \varepsilon < f(z) < f(x) + \varepsilon.$$

Thus, we fix $h \leq \delta$ and for such an $h > 0$ we know

$$h(f(x) - \varepsilon) < g(x + h) - g(x)$$

$$= \int_x^{x+h} f(t)dt < h(f(x) + \varepsilon),$$

and

$$h(f(x) - \varepsilon) < -(g(x - h) - g(x))$$

$$= \int_{x-h}^{x} f(t)dt < h(f(x) + \varepsilon).$$

Thus,

$$f(x) - \varepsilon < \frac{g(x \pm h) - g(x)}{\pm h} < f(x) + \varepsilon.$$

Letting $h \to 0^+$ we get

$$|g'(x) - f(x)| < \varepsilon.$$

However, $\varepsilon > 0$ was arbitrary. Thus we get

$$g'(x) = f(x).$$

Q.E.D.

Some consequences of this theorem:

Thus, we see that for $g(x) = \int_a^x f(t)dt$ we have $g'(x) = f(x)$. Now, let $F(x)$ be any antiderivative of $f(x)$. Since $g(x)$ is an antiderivative, we know that there exists a C so that

$$F(x) = g(x) + C.$$

Moreover, $g(a) = 0$, so

$$F(a) = C.$$

Also $\int_a^b f(x)dx = g(b)$, thus

$$F(b) = g(b) + C$$

$$= \int_a^b f(x)dx + F(a)$$

which implies that

$$F(b) - F(a) = \int_a^b f(x)dx.$$

Note that this holds for **any** antiderivative F of f. Let us now use this to calculate definite integrals.

Example: Calculate

$$\int_0^1 x^2 dx.$$

Here, $f(x) = x^2$, thus an antiderivative is $F(x) = \frac{x^3}{3}$. Hence

$$\int_0^1 x^2 dx = F(1) - F(0)$$

$$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$