The Shrinking Of A Uniformly Convex Curve To A Point Under The Curve Shortening Flow And Its Longterm Behavior: A Discussion Of The Gage-Hamilton Theorem

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INTRODUCTION:

In this paper we consider the flow generated by the following differential equation, where γ_0 is a convex closed curve embedded in R^2:

$$\begin{aligned} \frac{\partial \gamma}{\partial t} &= k\vec{n} \\ \gamma(p,0) &= \gamma_0(p) \end{aligned} \tag{1}$$

where we assume $p \in [0,P]$ and $t \in [0,T)$ for some $T > 0$.

Here, k denotes the curvature of γ, $\vec{n}$ is the inward unit normal to γ and $\vec{t}$ is the unit tangent to γ. Also, we assume that γ_0 is parameterized in the usual (counterclockwise) way.

For the sake of simplicity we assume that γ_0 is uniformly convex, and this is equivalent to assuming k_0, the curvature of γ_0, is positive.

What we intend to prove is that under such assumptions on γ_0, the solution γ is uniformly convex, shrinks to a point in finite time, and after normalizing the inclosed area to equal π, we have γ converges to the unit circle.

AN OUTLINE OF THE PAPER:

First we will establish a parabolic model to show existence and regularity of solutions to (1) for small time, assuming sufficient regularity of our initial data. Assuming we have a C^2 solution to (1) for small time, we then will get that $|k| < C$ for some constant C. Moreover, it also follows that $k(\cdot, t) > 0$ for small time t, because $inf k(\cdot, 0) > 0$. Then we will show that $0 < k < C$ for small time implies that $\gamma(\cdot, t)$ remains embedded, assuming γ_0 is embedded. This then allows us to parameterize $\gamma(\cdot, t)$ by θ. Once again, assuming the existence and regularity (C^4 at least) of a solution to (1), one then can show that the curvature satisfies the equation $k_\tau = k^2 k_{\theta\theta} + k^3$, where here ∂_τ is the partial with respect to time, holding θ fixed. Solutions of this PDE have the property that $k(\cdot, 0) > 0 \Rightarrow k$ remains positive. We then define h the support function of γ. We then establish that we can recover γ from h, and that h satisfies a parabolic PDE. This then gives us uniqueness of solutions to (1). We then give a representation of γ in terms of k. Lastly, we establish that our solution shrinks to a round point in finite time. We do this by first showing the enclosed area $A(t)$ of $\gamma(\cdot, t)$ must tend to zero in finite time, namely at $\frac{A(0)}{2\pi}$. Then we examine the rescaled problem so that the enclosed area is always π, and establish that the solution of this normalized problem tends to the unit circle in finite time.

RELATION BETWEEN HEAT EQUATION AND (1):

Let $s = s(t)$ be an arc-length element of $\gamma(\cdot, t)$, i.e. $|\gamma_s| = 1$. Then by the Frenet formulas ($\vec{t}_p = |\gamma_p| k \vec{n}$ and $\vec{n}_p = -|\gamma_p| k \vec{t}$), assuming γ satisfies (1), we obtain that $\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial s^2}$. This is a nonlinear parabolic second order system of 2 equations (because $\gamma \in R^2$). Namely, $\frac{\partial \gamma}{\partial t} = \frac{1}{|\gamma_p|^2} \frac{\partial^2 \gamma}{\partial p^2} - \frac{1}{|\gamma_p|^4} \frac{\partial \gamma}{\partial p} < \gamma_{pp}, \gamma_p >$.

A PARABOLIC MODEL FOR (1):

Suppose that $\gamma(p, t)$ is a solution to (1), with $p \in [0, P]$ as our parameter. Note that any curve $\tilde{\gamma}(p, t) := \gamma(\varphi(p), t)$ with $\varphi_p > 0$ is also a solution to (1). This is true since $\tilde{\gamma}_t(p, t) = \gamma_t(\varphi(p), t) = k(\varphi(p), t)\vec{n}(\varphi(p), t) =$ (since k is parameter invariant as one can easily check) $\tilde{k}(p, t)\tilde{\vec{n}}(p, t)$.

For this discussion we will use $p \in S^1$ is our parameter, and we will assume that q is the arclength parameter for our initial curve γ_0.

For the following discussion we will assume that our initial curve $\Gamma := \gamma_0$ is smooth. We then want to obtain a single parabolic equation from (1) (which is inherently a parabolic system) by viewing γ as a graph over Γ. Lastly, without loss of generality, we will assume our curve Γ has length 2π.

For all t fixed, let $\psi(\cdot,t) : S^1 \to S^1$ be a smooth diffeomorphism (still to be determined). Also we will require $\psi_q(\cdot,t) > 0$ for each t fixed. We will use the notation $p = p(q,t) = \psi(q,t)$, and we write $\gamma(p,t) = \Gamma(q) + d(q,t)\vec{N}(q)$. Here $\vec{N}$ is the inward unit normal of Γ. Also, we let K denote the curvature of Γ. Taking a derivative of $\gamma(p,t)$ with respect to q we get:

$$\gamma_q(p,t)\psi_q(q,t) = (1 - d(q,t)K(q))\Gamma_q(q) + d_q(q,t)\vec{N}(q).$$

Taking another derivative with respect to q we then get:
$\gamma_{qq}(p,t)\psi_q^2(q,t) + \gamma_q(p,t)\psi_{qq}(q,t) =$
$\{K(q) - d(q,t)K^2(q) + d_{qq}(q,t)\}\vec{N}(q) - \{2d_q(q,t)K(q) + d(q,t)K_q(q)\}\Gamma_q(q).$

We then get (using the notation $\Gamma_q = \vec{T} = (T_1, T_2)$):
$\gamma_q(p,t) = \frac{1}{\psi_q}\{(1 - dK)\Gamma_q + d_q\vec{N}\}|_{(q,t)} =$
$(\frac{1-dK}{\psi_q}T_1 - \frac{d_q}{\psi_q}T_2, \frac{d_q}{\psi_q}T_1 + \frac{1-dK}{\psi_q}T_2)|_{(q,t)}$, and $|\gamma_q(p,t)|^2 = \{\frac{(1-Kd)^2}{\psi_q^2} + \frac{d_q^2}{\psi_q^2}\}|_{(q,t)}$.
Also,
$\gamma_{qq}(p,t) = \{[\frac{K-K^2d+d_{qq}}{\psi_q^2} - \frac{\psi_{qq}d_q}{\psi_q^3}]\vec{N} - [\frac{2d_qK+dK_q}{\psi_q^2} + \frac{\psi_{qq}(1-Kd)}{\psi_q^3}]\Gamma_q\}|_{(q,t)} =$
$(A\vec{N} - B\Gamma_q)|_{(q,t)} = (-BT_1 - AT_2, AT_1 - BT_2)|_{(q,t)}$.

Therefore, $k(p,t) = \psi_q^3\frac{(AT_1-BT_2)(\frac{1-Kd}{\psi_q}T_1-\frac{d_q}{\psi_q}T_2)+(BT_1+AT_2)(\frac{d_q}{\psi_q}T_1+\frac{1-Kd}{\psi_q}T_2)}{((1-Kd)^2+d_q^2)^{\frac{3}{2}}}|_{(q,t)} =$
$\frac{\psi_q^2(A(1-Kd)+Bd_q)}{((1-Kd)^2+d_q^2)^{\frac{3}{2}}}|_{(q,t)} = \frac{(1-Kd)(K(1-Kd)+d_{qq})+d_q(2d_qK+dK_q)}{((1-Kd)^2+d_q^2)^{\frac{3}{2}}}|_{(q,t)}$.

Now, $\gamma(p,t) = \gamma(\psi(q,t),t) = \Gamma(q) + d(q,t)\vec{N}(q)$. Thus,
$\gamma_q(\psi(q,t),t)\psi_t(q,t) + \gamma_t(\psi(q,t),t) = \gamma_q(p,t)\psi_t(q,t) + k(p,t)\vec{n}(p,t) =$
$d_t(q,t)\vec{N}(q)$. Writing out what the left hand side is in terms of our above calculations we then get:

$d_t(q,t)\vec{N}(q) = \{\{(1-Kd)\Gamma_q + d_q\vec{N}\}\frac{\psi_t}{\psi_q} + \frac{(1-Kd)(K(1-Kd)+d_{qq})+d_q(2d_qK+dK_q)}{((1-Kd)^2+d_q^2)^{\frac{3}{2}}}\{\frac{1-Kd}{\sqrt{(1-Kd)^2+d_q^2}}\vec{N} - \frac{d_q}{\sqrt{(1-Kd)^2+d_q^2}}\Gamma_q\}\}|_{(q,t)}$.

Because the Γ_q component of the right hand side of the above equation must be zero, we then get:
$\frac{\psi_t}{\psi_q}|_{(q,t)} = \frac{kd_q}{(1-Kd)\sqrt{(1-Kd)^2+d_q^2}}|_{(q,t)} = \frac{d_q(1-Kd)(K-K^2d+d_{qq})+d_q^2(2d_qK+dK_q)}{(1-Kd)((1-Kd)^2+d_q^2)^2}|_{(q,t)} := f(q,t)$.

Also, comparing the $\vec{N}$ components we see:
$d_t(q,t) = \{d_q\frac{\psi_t}{\psi_q} + \frac{k(1-Kd)}{\sqrt{(1-Kd)^2+d_q^2}}\}|_{(q,t)}$
$= \{\frac{d_q^2(1-Kd)(K-K^2d+d_{qq})+d_q^3(2d_qK+dK_q)}{(1-Kd)((1-Kd)^2+d_q^2)^2} + \frac{(1-Kd)^2(K-K^2d+d_{qq})+d_q(1-Kd)(2d_qK+dK_q)}{((1-Kd)^2+d_q^2)^2}\}|_{(q,t)}$.

Rewriting the right hand side we get the following:

$$\begin{aligned} d_t &= \frac{d_{qq}}{(1-Kd)^2+d_q^2} + \frac{K(1-Kd)^2+d_q(2d_qK+dK_q)}{(1-Kd)((1-Kd)^2+d_q^2)} \\ d|_{t=0} &= 0 \end{aligned} \quad (2)$$

This is quasilinear and strictly parabolic as long as $(1-Kd)^2+d_q^2 > 0$. We can invoke the standard theory of 2rd order quasilinear parabolic equations (see [6] for instance) to establish existence and regularity of d for small time. Moreover, we then get $|k|$ is bounded and $k > 0$ for small time, a fact which is needed to prove that γ remains embedded, assuming $\Gamma = \gamma_0$ is embedded and $k_0 = k(\cdot, 0) > 0$.

However, to actually use this to establish a short term existence and regularity of a solution to (1), we also need to solve the following PDE for a smooth diffeomorphism $\psi(\cdot,t) : S^1 \to S^1$ with $\psi_q(\cdot,t) > 0$ for each t fixed.

$$\begin{aligned} \psi_t(q,t) &= \psi_q(q,t)f(q,t) \\ \psi(q,0) &= q \\ \psi_q > 0 & \end{aligned} \quad (3)$$

Thus we get the existence of a solution γ for small time t, moreover this solution remains closed and inherents regularity from d.

THE ISSUE OF REPARAMETERIZING $\gamma(\cdot, t)$:

Consider

$$\frac{\partial \gamma}{\partial t} = k\vec{n} + G(\gamma, \theta, k)\vec{t} \tag{4}$$

where $\vec{t}$ is the unit tangent, θ the angle such that the unit tangent vector $\vec{t}$ $= (\cos\theta, \sin\theta)$ and G is a smooth function 2π-periodic in θ.

Claim: If γ is a solution of (4) which is in C^∞, then $\exists \phi$ with $\phi(\cdot, t) : [0, P] \to [0, P]$ a C^1 diffeomorphism for each t fixed , which also satisfies $\phi_p > 0$, $\phi(p, 0) = p$ and $\tilde{\gamma}(p, t) = \gamma(\phi(p, t), t)$ solves (1).

Proof: $\frac{\partial \tilde{\gamma}}{\partial t}(p, t) = \frac{\partial \gamma}{\partial t}(\phi(p, t), t) + \frac{\partial \gamma}{\partial p}(\phi(p, t), t)\frac{\partial \phi}{\partial t}(p, t)$
$= k(\phi(p, t), t)\vec{n}(\phi(p, t), t) + G(\gamma(\phi(p, t), t), \theta(\gamma(\phi(p, t), t)), k(\phi(p, t), t))\vec{t} +$
$\frac{\partial \phi}{\partial t}(p, t)\gamma_p(\phi(p, t), t)$.

But then, all we need is $\frac{\partial \phi}{\partial t}(p, t) =$
$-|\gamma_p(\phi(p, t), t)|^{-1}G(\gamma(\phi(p, t), t), \theta(\gamma(\phi(p, t), t)), k(\phi(p, t), t))$. This is an O.D.E. where p is a parameter. From the smooth dependence on a parameter for solutions of O.D.E.'s we can deduce the existence of a solution satisfying our desired properties.
Q.E.D.

Thus, the geometry of (4) depends only on its normal velocity k, while the tangent velocity merely alters the parameterization of the flow.

PRESERVING EMBEDEDNESS:

Lemma: Let $f : [0, P] \times [0, P] \times [0, T) \to R$,
$f(p_1, p_2, t) := |\gamma(p_1, t) - \gamma(p_2, t)|^2$. Then $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial s_1^2} + \frac{\partial^2 f}{\partial s_2^2} - 4$, where s_1 and s_2 are arclength elements with respect to the variables p_1 and p_2.

note: The idea is to show that the function f is positive, provided $f|_{t=0} > 0$.

Proof: $f(p_1, p_2, t) = < \gamma(p_1, t) - \gamma(p_2, t), \gamma(p_1, t) - \gamma(p_2, t) >$.

Therefore, $\frac{\partial f}{\partial t}(p_1, p_2, t) =$
$2 < \gamma(p_1, t) - \gamma(p_2, t), k(p_1, t)\vec{n}(p_1, t) - k(p_2, t)\vec{n}(p_2, t) >$,
$\frac{\partial f}{\partial s_i}(p_1, p_2, t) = 2 < \gamma(p_1, t) - \gamma(p_2, t), (-1)^{i-1}\vec{t}(p_i, t) >$,
$\frac{\partial^2 f}{\partial s_i^2}(p_1, p_2, t)$
$= 2 < \vec{t}(p_i, t), \vec{t}(p_i, t) > + 2 < \gamma(p_1, t) - \gamma(p_2, t), (-1)^{i-1}k(p_i, t)\vec{n}(p_i, t) >$
$= 2 + 2 < \gamma(p_1, t) - \gamma(p_2, t), (-1)^{i-1}k(p_i, t)\vec{n}(p_i, t) >$,
for $i = 1, 2$.
Q.E.D.

note: $\frac{\partial}{\partial s_i} = \frac{1}{|\gamma_{p_i}|}\frac{\partial}{\partial p_i}$ and $\frac{\partial^2}{\partial s_i^2} = \frac{1}{|\gamma_{p_i}|^2}\frac{\partial^2}{\partial p_i^2} - \frac{<\gamma_{p_i p_i}, \gamma_{p_i}>}{|\gamma_{p_i}|^4}\frac{\partial}{\partial p_i}$.

Lemma: Let $f, g : [0, L] \to R^2$ be curves parameterized by arc-length. Let A, B be the endpoints of g and C, D the endpoints of f. Assume that g along with $\vec{AB}$ is the boundary of a convex set, f, g have continuous tangents and piecewise continuous curvatures, and that the curve g is transversed in the counterclockwise sense so that its curvature is positive. Lastly assume $k_g \geq |k_f|$. Then $d(A, B) \leq d(C, D)$.

Proof: Orient our curves so that $\vec{AB}$ and $\vec{CD}$ lie on the x-axis. We use the arclength elements for both curves as our parameter.

Let $\theta_g(s)$ be the angle the tangent line of g at s makes with the x-axis, and let $\theta_f(s)$ be the angle the tangent line of f at s makes with the x-axis.

Since $k_g > 0$, $\exists$ exactly one point s_0 so that the tangent line of g at s_0 is parallel to the x-axis (i.e. $\theta_g = 0$).

$\frac{d\theta_g}{ds} = k_g \geq |k_f| = |\frac{d\theta_f}{ds}|$ $(*)$.

Below we use the notation $f = (f_1, f_2)$, $\vec{t^f} = (-\cos\theta_f, \sin\theta_f)$, $g = (g_1, g_2)$ and $\vec{t^g} = (-\cos\theta_g, \sin\theta_g)$.

Thus, after integration of both sides of $(*)$ we get $|\theta_f(s) - \theta_f(s_0)| \leq |\theta_g(s)|$, and $|\theta_g(s)| \leq \pi$ since g is convex.

Therefore, $d(C, D)$
$= f_1(0) - f_1(L)$
$\geq \cos\theta_f(s_0)(f_1(0) - f_1(L))$
$= \cos\theta_f(s_0)(f_1(0) - f_1(L)) + \sin\theta_f(s_0)(f_2(L) - f_2(0))$
$=$ (by the fund. thm. of calc.)
$\cos(\theta_f(s_0)) \int_0^L -\frac{\partial}{\partial s} f_1(s) ds + \sin(\theta_f(s_0)) \int_0^L \frac{\partial}{\partial s} f_2(s) ds$
$= \cos\theta_f(s_0) \int_0^L \cos\theta_f(s) ds + \sin\theta_f(s_0) \int_0^L \sin\theta_f(s) ds$
$= \int_0^L \cos(\theta_f(s) - \theta_f(s_0)) ds$
$= \int_0^L \cos|\theta_f(s) - \theta_f(s_0)| ds$
$\geq \int_0^L \cos|\theta_g(s)| ds$
$= \int_0^L \cos\theta_g(s) ds$
$=$ (by the fund. thm. of calc.) $\int_0^L -\frac{\partial}{\partial s} g_1(s) ds$
$= g_1(0) - g_1(L)$
$= d(A, B)$.
Q.E.D.

Corollary: $|k| \leq K \Rightarrow f(p_1, p_2, t) \geq (\frac{2}{K} sin(\frac{K}{2} s(p_1, p_2, t))^2$, where $s(p_1, p_2, t) = |\int_{p_1}^{p_2} |\gamma_p(p, t)| dp|$.

Proof: Using the notation of the above lemma, we let f be the portion of $\gamma(\cdot, t)$ between $\gamma(p_1, t)$ and $\gamma(p_2, t)$ (so not to be confused with $f(p_1, p_2, t)$, we will always use $|\gamma(p_1, t) - \gamma(p_2, t)|^2$ instead of $f(p_1, p_2, t)$ in this proof); let $L = s(p_1, p_2, t) := |\int_{p_1}^{p_2} |\gamma_p(p, t)| dp|$. Let g be the arc of length $s(p_1, p_2, t)$ of a circle of radius $\frac{1}{K}$ centered at the origin.

Then A, B will be the endpoints of g and $C = \gamma(p_1, t)$, $D = \gamma(p_2, t)$. Thus, $d(A, B)$ is the length of the side $\vec{AB}$ of the triangle whose vertices are the origin, A and B. The angle opposite $\vec{AB}$ is $Ks(p_1, p_2, t)$, and thus the length of $\vec{AB}$ is $2\frac{1}{K}\sin(\frac{Ks(p_1, p_2, t)}{2})$. Therefore, $|\gamma(p_1, t) - \gamma(p_2, t)| = d(C, D)$ $\geq d(A, B) = \frac{2}{K}\sin(\frac{Ks(p_1, p_2, t)}{2})$.
Q.E.D.

In the following theorem we assume that the curvature k is bounded from above and below. Thus will be true for at least small time. One can see this later when we establish that the curvature satisfies a certain 2rd order quasilinear parabolic equation. However, we can get this directly by the discussion given on short term existence (1) via viewing out solution γ as the graph over our initial curve γ_0

Theorem: Let $\gamma : [0,P] \times [0,T) \to R^2$, $\gamma(\cdot,t)$ a closed curve; $\frac{\partial\gamma}{\partial t} = k\vec{n}$. If $|k| < C$ and $\gamma(\cdot,0)$ is embedded, then $\gamma(\cdot,t)$ is embedded $\forall t \in [0,T)$.

Proof: Let $E := \{(p_1,p_2,t) : s(p_1,p_2,t) < \frac{\pi}{C}\}$.

First we demonstrate that on E $f = 0$ if and only if $p_1 = p_2$. By the corollary, $f = 0 \Rightarrow \sin(\frac{C}{2}s(p_1,p_2,t)) = 0 \Rightarrow s(p_1,p_2,t) = 0 \Rightarrow p_1 = p_2$. Conversely, $p_1 = p_2$ implies $f = 0$ by the definition of f.

Let $D := ([0,P] \times [0,P] \times [0,T)) \setminus E$. We will use a maximum principle argument to show the minimum of f is positive on D, which then proves the theorem.

$\partial D = A \cup B$, where $A := \{(p_1,p_2,t) : s(p_1,p_2,t) = \frac{\pi}{C}, t \in [0,T)\}$ and $B := \{(p_1,p_2,0) : s(p_1,p_2,0) \geq \frac{\pi}{C}\}$.

On A, $f \geq (\frac{2}{C})^2 > 0$ by the corollary, and on B $\gamma(\cdot,0)$ is embedded $\Rightarrow$ $f > 0$. Thus $f > 0$ on ∂D.

Let $m = min\{min_A f, min_B f\} > 0$.

Consider $g = f + \varepsilon t$ for $\varepsilon > 0$.

Then g satisfies $\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial s_1^2} + \frac{\partial^2 g}{\partial s_2^2} - 4 + \varepsilon$.

Let $0 < \delta < m$, and suppose g achieves the minimum value of $m - \delta$ on D. Let $t_0 = inf\{t : g(p_1,p_2,t) = m - \delta\}$. The continuity of g and compactness of D along with our boundary estimates imply that $m - \delta$ is achieved at an interior point $(\bar{p_1},\bar{p_2},\bar{t})$. We note here that $g_{s_1 s_2} = \frac{1}{|\gamma_{p_1}||\gamma_{p_2}|} g_{p_1 p_2}$ and for $i = 1,2$ $g_{s_i s_i} = \frac{1}{|\gamma_{p_i}|^2} g_{p_i p_i} - \frac{<\gamma_{p_i p_i}, \gamma_{p_i}>}{|\gamma_{p_i}|^4} g_{p_i}$.

At this minimum point $\frac{\partial g}{\partial t} \leq 0$, $g_{p_i} = 0$, and the Hessian matrix of g with

respect to the variables p_1 and p_2 is non-negative definite. This then implies that (at $(\bar{p_1}, \bar{p_2}, \bar{t})$) we have the following:
$g_{s_i s_i} = \frac{1}{|\gamma_{p_i}|^2} g_{p_i p_i}$ for $i = 1, 2$
$0 \leq \frac{1}{|\gamma_{p_1}|^2 |\gamma_{p_2}|^2}(g_{p_1 p_1} g_{p_2 p_2} - (g_{p_1 p_2})^2) = g_{s_1 s_1} g_{s_2 s_2} - (g_{s_1 s_2})^2.$

Also, by a direct calculation one sees that $\frac{\partial^2 g}{\partial s_1 \partial s_2} = -2 < \vec{t}(\bar{p_1}, \bar{t}), \vec{t}(\bar{p_2}, \bar{t}) > = \pm 2$ since at a minimum point the tangent lines to the curve at p_1 and p_2 must be parallel.

Lastly at this minimum point we also have $0 \leq (\sqrt{g_{s_1 s_1}} - \sqrt{g_{s_2 s_2}})^2 \Rightarrow \frac{\partial^2 g}{\partial s_1^2} + \frac{\partial^2 g}{\partial s_2^2} \geq 2\sqrt{\frac{\partial^2 g}{\partial s_1^2}\frac{\partial^2 g}{\partial s_2^2}} \geq 2|\frac{\partial^2 g}{\partial s_1 \partial s_2}| \geq 4$. But this is a contradiction.

δ is arbitrary $\Rightarrow g \geq m$ on $D \Rightarrow f \geq m - \varepsilon T$ on D. Letting $\varepsilon \to 0+ \Rightarrow f \geq m > 0$ on D.
Q.E.D.

MORE ON PRESERVING CONVEXITY:

For a given solution γ to (1), we have established that γ is embedded and convex for small time, and hence we may parameterize $\gamma(\cdot, t)$ by θ, where we assume θ such that $\vec{n} = -(\cos\theta, \sin\theta)$ or $\vec{t} = (\cos\theta, \sin\theta)$. In the following, we assume that our solution γ of (1) is at least C^4. To determine the evolution equation for curvature when using θ as our parameter, we take τ as the time parameter. Thus we have a change of variables (p, t) to (θ, τ). Even though $\tau = t$, we don't necessarily have $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t}$, because $\frac{\partial}{\partial t}$ is the derivative in t of a function assuming p is fixed, and $\frac{\partial}{\partial \tau}$ is the derivative in τ of a function assuming θ is fixed.

Lemma: $\frac{\partial k}{\partial \tau} = k^2 \frac{\partial^2 k}{\partial \theta^2} + k^3$.

Proof: First we show the following:
$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3$, where s is our arclength element so that $ds = |\gamma_p| dp$.

To show this we note the following claim:
$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + k^2 \frac{\partial}{\partial s}$

This is true because: $\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\frac{1}{|\gamma_p|}\frac{\partial}{\partial p} =$ (using $\frac{\partial|\gamma_p|}{\partial t} = -k^2|\gamma_p|$ which we will establish shortly) $\frac{k^2}{|\gamma_p|}\frac{\partial}{\partial p} + \frac{1}{|\gamma_p|}\frac{\partial}{\partial p}\frac{\partial}{\partial t} = k^2\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}$.

Note: We will also assume that $\vec{n}_t = -k_s\vec{t}$ which we will also establish shortly. However computing this directly from our definition of $\vec{n} = -(\cos\theta, \sin\theta)$ we get $\vec{n}_t = -(-\sin\theta, \cos\theta)\frac{\partial\theta}{\partial t} = -\vec{t}\theta_t$. Thus $\theta_t = k_s$. However, we also have $\frac{\partial\theta}{\partial s} = k$ in the same way using a direct calculation and the Frenet formulas.

Thus $k_t = \frac{\partial^2\theta}{\partial t\partial s} = \frac{\partial^2\theta}{\partial s\partial t} + k^2\frac{\partial\theta}{\partial s} = k_{ss} + k^3$.

Now we proceed to prove the lemma. By the chain rule we have:
$\frac{\partial k}{\partial t} = k_\tau + k_\theta\theta_t = k_\tau + k_\theta k_s = k_\tau + k(k_\theta)^2$;
Also $\frac{\partial}{\partial s} = k\frac{\partial}{\partial\theta} \Rightarrow k_{ss} = k(kk_\theta)_\theta = k(k_\theta)^2 + k^2k_{\theta\theta}$.

Putting this together, we get the lemma.
Q.E.D.

Theorem: Suppose k is a solution to:

$$k_\tau = k^2k_{\theta\theta} + k^3 \text{ for } \theta \in S^1,\ \tau \in [0, T]$$
$$k|_{\tau=0} = k_0 > 0 \text{ is given.}$$

Proof: Fix $0 < T < \omega$, where for $\tau \in [0, \omega)$ we have a solution. Pick $\lambda \geq 0$ such that $\lambda > sup_{[0,2\pi]\times[0,T]}k^2$. Suppose that $k \leq 0$ at some point in $[0, 2\pi] \times [0, T]$, then so is $v = e^{-\lambda\tau}k$. Therefore the minimum of v on $[0, 2\pi] \times [0, T]$ is non-positive. We let $(\bar{\theta}, \bar{\tau})$ be a minimum point of v on $[0, 2\pi] \times [0, T]$.

We then see that v satisfies the following PDE:

$$v_\tau = e^{-\lambda\tau}k_\tau - \lambda e^{-\lambda\tau}k = e^{-\lambda\tau}\{k^2k_{\theta\theta} + k^3\} - \lambda e^{-\lambda\tau}k = k^2v_{\theta\theta} + \{k^2 - \lambda\}v.$$

At $(\bar{\theta}, \bar{\tau})$ we have: $v_\tau(\bar{\theta}, \bar{\tau}) \leq 0$, $v_{\theta\theta}(\bar{\theta}, \bar{\tau}) \geq 0$ and $v(\bar{\theta}, \bar{\tau}) \leq 0$. But then since $k^2v_{\theta\theta} + \{k^2 - \lambda\}v \geq 0$ at $(\bar{\theta}, \bar{\tau})$, and this is strictly positive unless $v = 0$. Therefore we can conclude that $k \geq 0$ on $[0, 2\pi] \times [0, T]$. Now we want to show $k > 0$.

Let $m(\tau) = min_{\theta\in[0,2\pi]}k(\theta,\tau) = k(\theta(\tau),\tau)$ (for some $\theta(\tau) \in [0, 2\pi]$).

If $\theta(\tau)$ is C^1, then $\frac{dm}{d\tau}(\tau) = k_\theta(\theta(\tau),\tau)\frac{d\theta}{d\tau}(\tau) + k_\tau(\theta(\tau),\tau) =$ $\{k^2 k_{\theta\theta} + k^3\}|_{(\theta(\tau),\tau)} \leq k^3(\theta(\tau),\tau) = m^3(\tau)$. But, in general $\theta(\tau)$ may not be a C^1 function. So instead we look at $liminf_{h\to 0}\frac{m(\tau+h)-m(\tau)}{h}$, and get a similar result for this.

Thus,
$\frac{m(\tau+h)-m(\tau)}{h} = \frac{k(\theta(\tau+h),\tau+h)-k(\theta(\tau),\tau)}{h} \geq \frac{k(\theta(\tau+h),\tau+h)-k(\theta(\tau+h),\tau)}{h} =$
$k_\tau(\theta(\tau+h),\xi_{\tau,h}) = (k^2 k_{\theta\theta} + k^3)|_{(\theta(\tau+h),\xi_{\tau,h})}$, where $\xi_{\tau,h}$ lies strictly between τ and $\tau + h$.

Let $\{h_i\}_{i=1}^{\infty}$ be any sequence with limit zero such that $\frac{m(\tau+h_i)-m(\tau)}{h_i}$ has a limit, and let C denote this limit.

Then,
$\lim_{i\to\infty}\frac{m(\tau+h_i)-m(\tau)-h_i C}{h_i} = 0 \Rightarrow$
$\lim_{i\to\infty}\{m(\tau+h_i) - m(\tau) - h_i C\} = 0 \Rightarrow$
$\lim_{i\to\infty} m(\tau+h_i) = m(\tau)$, i.e. $\lim_{i\to\infty} k(\theta(\tau+h_i),\tau+h_i) = k(\theta(\tau),\tau)$.
$\{\theta(\tau+h_i)\}_{i=1}^{\infty}$ is a bounded sequence of numbers, therefore there exists a subsequence $\{h_{i_k}\}_{k=1}^{\infty}$ such that $\theta(\tau+h_{i_k}) \to \hat{\theta}$ as $k\to\infty$. Thus $m(\tau) =$ $\lim_{k\to\infty} m(\tau+h_{i_k}) = lim_{k\to\infty}k(\theta(\tau+h_{i_k}),\tau+h_{i_k}) = k(\hat{\theta},\tau)$.

Therefore,
$\lim_{i\to\infty}\frac{m(\tau+h_i)-m(\tau)}{h_i}$
$= \lim_{k\to\infty}\frac{m(\tau+h_{i_k})-m(\tau)}{h_{i_k}}$
$\geq \lim_{k\to\infty}\{k^2(\theta(\tau+h_{i_k}),\xi_{\tau,h_{i_k}})k_{\theta\theta}(\theta(\tau+h_{i_k}),\xi_{\tau,h_{i_k}}) + k^3(\theta(\tau+h_{i_k}),\xi_{\tau,h_{i_k}})\}$
$= k^2(\hat{\theta},\tau)k_{\theta\theta}(\hat{\theta},\tau) + k^3(\hat{\theta},\tau)$
$\geq (m(\tau))^3$
$\geq 0.$

Since for τ fixed, $k(\cdot,\tau)$ has a minimum at $\hat{\theta}$; thus $k_{\theta\theta}(\bar{\theta},\tau) \geq 0$. Therefore, $liminf_{h\to 0}\frac{m(\tau+h)-m(\tau)}{h} \geq (m(\tau))^3 \geq 0$.

Since $liminf_{h\to 0}\frac{m(\tau+h)-m(\tau)}{h} \geq (m(\tau))^3 \geq 0$ we have:
Given $\varepsilon > 0$ $\exists H > 0$ such that for $0 < h < H$ we have

$\frac{m(\tau+h)-m(\tau)}{h} \geq (m(\tau))^3 - \varepsilon \geq -\varepsilon$. Therefore, $m(\tau+h) \geq m(\tau) - \varepsilon h$.

Assuming $m(0) > 0$, let $\varepsilon = \frac{m(0)}{2}$ and h less than the minimum of 1 and H. We then get $m(h) \geq \frac{m(0)}{2} > 0$. Thus $k > 0$ for τ sufficiently small.
Q.E.D.

Note: Even though we have already established (for a solution to (1)) k remains positive, the above argument was included because it is useful at future points.

THE SUPPORT FUNCTION OF A CONVEX CURVE:

Given any curve γ with positive curvature (i.e. γ uniformly convex); $p \in I$. Let $\vec{n}(p) = -(\cos\theta, \sin\theta)$. Here $\theta = \theta(p)$ is called the normal angle at $\gamma(p)$. So then we also have $\vec{t}(p) = (-\sin\theta, \cos\theta)$ is the unit tangent. Let $J = \theta(I)$. If γ is a simple closed curve then $J = [0, 2\pi)$. For such curves we may use θ as our parameter (as is the case with $\gamma(\cdot, t)$ for small time t). Let $\gamma = \gamma(\theta)$ and $\vec{n} = -(\cos\theta, \sin\theta)$. The support function of γ defined in J is given by $h(\theta) =< \gamma(\theta), (\cos\theta, \sin\theta) >$. Writing $\gamma = (\gamma_1, \gamma_2)$, we have $h(\theta) = \gamma_1 \cos\theta + \gamma_2 \sin\theta$. Therefore, $h_\theta(\theta) = -\gamma_1 \sin\theta + \gamma_2 \cos\theta$. Thus γ can be recovered from the system:

$$\gamma_1 = h\cos\theta - h_\theta \sin\theta$$
$$\gamma_2 = h\sin\theta + h_\theta \cos\theta$$

Also, we see that $h_{\theta\theta} + h = < (\gamma^1_\theta, \gamma^2_\theta), \vec{t} > =$ (by the chain rule) $< \frac{d\gamma}{ds}\frac{ds}{d\theta}, \vec{t} >$ $= \frac{ds}{d\theta} = \frac{1}{k}$ (where our last equality can be seen via a direct calculation of $\frac{d}{ds}\vec{n}$ and what the Frenet formulas tell you).

The next computation proves quite useful when establishing the long term behavior of the flow. Here we assume $\gamma = \gamma(\cdot, t)$ (for small t) is our given solution to (1))

Using the notation $\frac{\partial}{\partial\tau}$ to be the time derivative when holding θ constant. We get that h satisfies:

$h_\tau = < \gamma_\tau, -\vec{n} > =$ (by the chain rule) $< \gamma_p p_\tau + \gamma_t, -\vec{n} > = < k\vec{n}, -\vec{n} > = -k$. Thus h satisfies the following parabolic PDE: $h_\tau = \frac{-1}{h_{\theta\theta}+h}$.

UNIQUENESS OF SOLUTIONS TO (1):

Assuming existence and C^4 regularity of a solution γ to (1), we have shown that the $\gamma(\cdot,t)$ remains embedded and convex. Also we have just shown that our solution γ can be recovered from its support function h, and h satisfies $h_\tau = \frac{-1}{h_{\theta\theta}+h}$. This equation is parabolic, and thus we will make use of this to deduce the uniqueness of solutions to (1).

Suppose that γ and $\tilde{\gamma}$ are two solutions to (1). Let k and $\tilde{k}$ be their respective curvatures, and h, $\tilde{h}$ their respective support functions. We define $w := h - \tilde{h}$. Then w satisfies $w_\tau = h_\tau - \tilde{h}_\tau = \frac{1}{\tilde{h}_{\theta\theta}+\tilde{h}} - \frac{1}{h_{\theta\theta}+h} = \frac{1}{(h_{\theta\theta}+h)(\tilde{h}_{\theta\theta}+\tilde{h})}\{h_{\theta\theta} - \tilde{h}_{\theta\theta} + h - \tilde{h}\} = k\tilde{k}\{w_{\theta\theta}+w\}$.

Therefore, w satisfies:

$$w_\tau = k\tilde{k}\{w_{\theta\theta}+w\}$$
$$w|_{\tau=0} = 0$$

For the following we let $A := k\tilde{k}$, and let $f(\tau) := \int_0^{2\pi} w^2(\theta,\tau)d\theta$. Then
$f'(\tau) = \frac{1}{2}\int_0^{2\pi} ww_\tau d\theta$
$= \frac{1}{2}\int Aww_{\theta\theta} + Aw^2 d\theta$
$=$ (by integration by parts) $-\frac{1}{2}\int[Aw_\theta + A_\theta w]w_\theta d\theta + \frac{1}{2}\int Aw^2 d\theta$
$\leq \frac{-infA}{2}\int w_\theta^2 d\theta + \frac{sup|A_\theta|}{2}\int |w||w_\theta|d\theta + \frac{supA}{2}\int w^2 d\theta$
Using Cauchy's inequality for $\varepsilon > 0$ (for $a, b, \varepsilon > 0$, $ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$), with $\varepsilon = \frac{sup|A_\theta|}{infA}$ we then get $f'(\tau) \leq \{\frac{(sup|A_\theta|)^2}{2infA} + \frac{supA}{2}\}\int w^2 d\theta = \{\frac{(sup|A_\theta|)^2}{2infA} + \frac{supA}{2}\}f(\tau)$. Thus by Gronwall's inequality we deduce that $f \equiv 0$, and thus $w \equiv 0$, and hence uniqueness of solutions to (1).

CONVEX CURVES IN THE PLANE:

Here we assume that our parameter θ is the angle such that

$\vec{t} = (\cos\theta, \sin\theta)$.

Lemma: A positive 2π-periodic function k represents the curvature of a simple closed uniformly convex C^2 plane curve $\Leftrightarrow \int_0^{2\pi} \frac{\cos\theta}{k(\theta)} d\theta = \int_0^{2\pi} \frac{\sin\theta}{k(\theta)} d\theta = 0$.

Proof: If $k > 0$ is the curvature of a closed simple curve with length L, then $\int_0^L \vec{t} ds = \int_0^L \gamma_s ds = \gamma(s)|_0^L = (0,0)$. But $\int_0^L \vec{t} ds = \int_0^L (\cos\theta(s), \sin\theta(s)) ds$. Thus, $\int_0^L \cos\theta(s) ds = \int_0^L \sin\theta(s) ds = 0$. Moreover, $\int_0^L \cos\theta(s) ds = \int_0^{2\pi} \frac{\cos\theta}{k(\theta)} d\theta$ and $\int_0^L \sin\theta(s) ds = \int_0^{2\pi} \frac{\sin\theta}{k(\theta)} d\theta$.

Conversely, assuming $k > 0$, $\int_0^{2\pi} \frac{\cos\theta}{k(\theta)} d\theta = 0$ and $\int_0^{2\pi} \frac{\sin\theta}{k(\theta)} d\theta = 0$, we define $\gamma(\theta) = (x(\theta), y(\theta))$ by $x(\theta) = \int_0^\theta \frac{\cos r}{k(r)} dr$ and $y(\theta) = \int_0^\theta \frac{\sin r}{k(r)} dr$. Clearly by our assumption γ is closed.

By the fundamental theorem of calculus, $\gamma_\theta = (\frac{\cos\theta}{k(\theta)}, \frac{\sin\theta}{k(\theta)})$. Hence, $\gamma_{\theta\theta} = \frac{1}{k^2}(-\sin\theta \cdot k - \cos\theta \cdot k_\theta, \cos\theta \cdot k - \sin\theta \cdot k_\theta)$. Thus the curvature of γ is given by $\frac{\gamma^2_{\theta\theta}\gamma^1_\theta - \gamma^1_{\theta\theta}\gamma^2_\theta}{|\gamma_\theta|^3} = k$.

To show that it is simple note that the map $\theta \mapsto \vec{t} = (\cos\theta, \sin\theta)$ is one-to-one. This insures that our curve is simple.
Q.E.D.

A REPRESENTATION FORMULA FOR OUR SOLUTION:

In the following discussion we give a representation formula for solutions to (1) by finding a unique 2π-periodic in θ solution to:

$$\begin{aligned} k_\tau &= k^2 k_{\theta\theta} + k^3 \\ k(\theta, 0) &= k_0(\theta) \end{aligned} \qquad (5)$$

for short term τ. (Here $\frac{\partial}{\partial\tau}$ is the partial with respect to time holding θ fixed and $\frac{\partial}{\partial\theta}$ is the partial with respect to our parameter holding time fixed).

It was shown that if $k(\cdot, 0) > 0$ then k, which satisfies this equation, remains positive. Moreover, it was also shown that the solution to (1) also satisfies this PDE. Thus starting with this PDE (namely (5)), with a positive initial value assumption, we may construct the unique solution to (1) from the solution to (5).

Assuming the existence and uniqueness of a solution to (5) with $k > 0$ smooth, we define $\gamma = (x, y)$ by $x(\theta, \tau) = \int_0^\theta \frac{\cos z}{k(z,\tau)}dz - \int_0^\tau k_\theta(0, r)dr + x(0, 0)$ and $y(\theta, \tau) = \int_0^\theta \frac{\sin z}{k(z,\tau)}dz + \int_0^\tau k(0, r)dr + y(0, 0)$. We are assuming that γ_0 was specified. Then this will give us a family of curves $\gamma(\cdot, \tau)$ such that $\gamma(\cdot, 0) = \gamma_0$ and γ is smooth. Let $\theta \in [0, 2\pi]$, and let $s(\theta)$ is the length of the piece of the curve with endpoints $\gamma(\theta, 0)$ and $\gamma(0, 0)$. Then $\gamma(\theta, 0) - \gamma(0, 0)$ = (by the fundamental theorem of calculus) $\int_0^{s(\theta)} \vec{t}ds = \int_0^\theta \frac{(\cos z, \sin z)}{k_0(z)}dz$.

Now we will establish that $\gamma_\tau = k\vec{n} - k_\theta\vec{t}$, where here we take θ to be so that $\vec{t} = (\cos\theta, \sin\theta)$ and $\vec{n} = (-\sin\theta, \cos\theta)$. We then define $\tilde{\gamma}(\theta, t) = \gamma(\phi(\theta, t), t)$ for some ϕ, with $\phi_\theta > 0$ and $\phi(\cdot, t) : [0, 2\pi] \to [0, 2\pi]$ a diffeomorphism for each fixed t, such that $\tilde{\gamma}_\tau = k\vec{n}$.

Note that for each fixed t, γ as defined is just a translation of $(\int_0^\theta \frac{\cos z}{k(z,\tau)}dz, \int_0^\theta \frac{\sin z}{k(z,\tau)}dz)$, which we have already established has curvature k. Thus γ has curvature k.

Now we compute x_τ and y_τ:

$x_\tau(\theta, \tau) = \int_0^\theta \frac{\partial}{\partial\tau}(\frac{1}{k(z,\tau)}) \cos z dz - k_\theta(0, \tau)$
$= \int_0^\theta \frac{-k_\tau(z,\tau)}{k^2(z,\tau)} \cos z dz - k_\theta(0, \tau)$
$= \int_0^\theta - \cos z(k_{zz} + k)dz - k_\theta(0, \tau)$
$= \int_0^\theta - \cos z k_{zz} dz - \int_0^\theta \cos z k dz - k_\theta(0, \tau)$
= (by integration by parts)
$\int_0^\theta - \sin z k_z dz - \cos z k_\theta|_{z=0}^{z=\theta} - \int_0^\theta \cos z k dz - k_\theta(0, \tau)$
= (by integration by parts)
$\int_0^\theta \cos z k dz - \sin z k|_{z=0}^{z=\theta} - \cos z k_\theta|_{z=0}^{z=\theta} - \int_0^\theta \cos z k dz - k_\theta(0, \tau)$
$= - \sin\theta k(\theta, \tau) - \cos\theta k_\theta(\theta, \tau)$.

$y_\tau(\theta, \tau) = \int_0^\theta \frac{\partial}{\partial\tau}(\frac{1}{k(z,\tau)}) \sin z dz + k(0, \tau)$
$= \int_0^\theta \frac{-k_\tau}{k^2} \sin z dz + k(0, \tau)$
$= \int_0^\theta - \sin z(k_{zz} + k)dz + k(0, \tau)$

$= \int_0^\theta -\sin z k_{zz} dz - \int_0^\theta \sin z k dz + k(0,\tau)$
$=$ (by integration by parts) $\int_0^\theta \cos z k_z dz - \sin z k_\theta|_{z=0}^{z=\theta} - \int_0^\theta \sin z k dz + k(0,\tau)$
$=$ (by integration by parts)
$\int_0^\theta \sin z k dz - \sin z k_\theta|_{z=0}^{z=\theta} + \cos z k|_{z=0}^{z=\theta} - \int_0^\theta \sin z k dz + k(0,\tau)$
$= -\sin\theta k_\theta(\theta,\tau) + \cos\theta k(\theta,\tau)$.

Thus $\gamma_\tau = (-\sin\theta, \cos\theta)k(\theta,\tau) - (\cos\theta, \sin\theta)k_\theta(\theta,\tau) = k\vec{n} - k_\theta\vec{t}$. Thus after an appropriate reparameterization we have a solution to (1).

Now we establish the uniqueness of a solution to (5).

Suppose there exists two solutions u, v to (5) with the same initial condition f, and let $w = u - v$. Then w satisfies:
$w_\tau = u^2 w_{\theta\theta} + \{(u+v)v_{\theta\theta} + u^2 + uv + v^2\}w = u^2 w_{\theta\theta} + bw$ subject to $w(\cdot, 0) \equiv 0$. We want to show $w \equiv 0$.

Let $g(\tau) := \int_0^{2\pi} w^2(\theta,\tau) d\theta$. Then
$\frac{1}{2}g'(\tau) = \int_0^{2\pi} w w_\tau d\theta$
$= \int_0^{2\pi} u^2 w w_{\theta\theta} + bw^2 d\theta$
$=$ (by integration by parts) $-2\int_0^{2\pi} u u_\theta w w_\theta d\theta - \int_0^{2\pi} u^2 w_\theta^2 d\theta + \int_0^{2\pi} bw^2 d\theta$
$\leq 2sup(u)sup|u_\theta| \int_0^{2\pi} |w||w_\theta| d\theta - inf(u^2) \int_0^{2\pi} w_\theta^2 d\theta + sup|b| \int_0^{2\pi} w^2 d\theta$
$= (*)$

Also, for any $\varepsilon > 0$, $\int |w||w_\theta| d\theta \leq \varepsilon \int w^2 d\theta + \frac{1}{\varepsilon}\int w_\theta^2 d\theta$ for any $\varepsilon > 0$. Using $\varepsilon = \frac{2sup(u)sup|u_\theta|}{inf(u^2)}$ we then get
$(*) \leq (\frac{(2sup(u)sup|u_\theta|)^2}{inf(u^2)} + sup|b|) \int_0^{2\pi} w^2 d\theta = C\int_0^{2\pi} w^2 d\theta$.

Therefore, $g'(\tau) \leq 2Cg(\tau)$. Integrating both sides from 0 to τ we get $g(\tau) \leq 2C\int_0^\tau g(t)dt \Rightarrow$ (by Gronwall's inequality) $g \equiv 0 \Rightarrow w \equiv 0$ and thus uniqueness is established.

To establish the existence and regularity of a periodic solution to (5), we refer the reader to M.Taylor chapter 15 sections 7 and 8 on quasilinear equations, or Lieberman's paper "The First Initial-Boundary Value Problem for Quasilinear Second Order Parabolic Equations."

EVOLUTION OF GEOMETRIC QUANTITIES:

Here we assume the Frenet formulas: $\frac{\partial \vec{t}}{\partial p} = |\gamma_p| k\vec{n}$ and $\frac{\partial \vec{n}}{\partial p} = -|\gamma_p|k\vec{t} = -k\gamma_p$. The length of $\gamma(\cdot, t)$ is given by $L(t) = \int_{P_0}^{P_1} |\gamma_p|dp$,

where our parameter $p \in [P_0, P_1]$ and s is an arc-length element (i.e. $ds = |\gamma_p|dp$). We now compute $\frac{dL}{dt}$.

$$
\begin{aligned}
\frac{dL}{dt}(t) &= \int_{P_0}^{P_1} \frac{\partial\sqrt{<\gamma_p,\gamma_p>}}{\partial t} dp \\
&= \int_{P_0}^{P_1} \frac{<\gamma_p,(k\vec{n})_p>}{|\gamma_p|} dp \\
&= \int_{P_0}^{P_1} \frac{<\gamma_p,(k\vec{n})_s|\gamma_p|>}{|\gamma_p|} dp \\
&= \int_{P_0}^{P_1} \frac{<\gamma_p,k_s|\gamma_p|\vec{n}-k^2|\gamma_p|\vec{t}>}{|\gamma_p|} dp \\
&= \int_{P_0}^{P_1} < \gamma_p, -k^2\vec{t} > dp \\
&= -\int_{P_0}^{P_1} k^2|\gamma_p|dp < 0.
\end{aligned}
$$

Therefore, $\frac{dL}{dt}$ is negative $\Rightarrow L(t)$ is a strictly decreasing function of time.

We note here two important observations used in the above calculation of $\frac{dL}{dt}$:

$$
\begin{aligned}
\frac{\partial|\gamma_p|}{\partial t} &= \frac{\partial}{\partial t}(< \gamma_p, \gamma_p >)^{\frac{1}{2}} \\
&= \frac{<\gamma_p,\gamma_{pt}>}{|\gamma_p|} \\
&= \frac{<\gamma_p,\gamma_{tp}>}{|\gamma_p|} \\
&= \frac{<\gamma_p,(k\vec{n})_p>}{|\gamma_p|} \\
&= \frac{<\gamma_p,k_p\vec{n}+k\vec{n_p}>}{|\gamma_p|} \\
&= \frac{<\gamma_p,kn_s|\gamma_p|>}{|\gamma_p|} \\
&= < \gamma_p, kn_s > \\
&= < \gamma_p, -k^2\vec{t} > \\
&= -k^2|\gamma_p|,
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_{pt} &= \gamma_{tp} \\
&= (k\vec{n})_p \\
&= k_p\vec{n} + k\vec{n_s}|\gamma_p|
\end{aligned}
$$

$= k_p \vec{n} - k^2 \gamma_p$
$= \frac{k_p}{|\gamma_p|}(-\gamma_p^2, \gamma_p^1) - k^2(\gamma_p^1, \gamma_p^2)$
$= (\frac{-k_p}{|\gamma_p|}\gamma_p^2 - k^2\gamma_p^1, \frac{k_p}{|\gamma_p|}\gamma_p^1 - k^2\gamma_p^2)$.

From these two observations it follows that, $\vec{n_t}$
$= \partial_t\{\frac{1}{|\gamma_p|}(-\gamma_p^2, \gamma_p^1)\}$
$= \frac{k^2}{|\gamma_p|}(-\gamma_p^2, \gamma_p^1) + \frac{1}{|\gamma_p|}(-\gamma_{pt}^2, \gamma_{pt}^1)$
$= \frac{k^2}{|\gamma_p|}(-\gamma_p^2, \gamma_p^1) + \frac{1}{|\gamma_p|}(-k_p\frac{\gamma_p^1}{|\gamma_p|} + k^2\gamma_p^2, -k_p\frac{\gamma_p^2}{|\gamma_p|} - k^2\gamma_p^1)$
$= -\frac{k_p}{|\gamma_p|^2}\gamma_p$
$= -k_s\vec{t}$.

Next we note that the area enclosed by $\gamma(\cdot, t)$ is given by:
$A(t) = \frac{1}{2}\int(\gamma^1\gamma_p^2 - \gamma^2\gamma_p^1)dp = -\frac{1}{2}\int_{\gamma(\cdot,t)} < \gamma, \vec{n} > ds$

Therefore,
$A'(t) = -\frac{1}{2}\int \frac{\partial(<\gamma,\vec{n}>|\gamma_p|)}{\partial t}dp$
$= \frac{1}{2}\int k^2 < \gamma, \vec{n} > |\gamma_p|dp - \frac{1}{2}\int \frac{\partial <\gamma,\vec{n}>}{\partial t}|\gamma_p|dp$
$= \frac{1}{2}\int k^2 < \gamma, \vec{n} > |\gamma_p|dp - \frac{1}{2}\int < \gamma_t, \vec{n} > |\gamma_p|dp - \frac{1}{2}\int < \gamma, \vec{n_t} > |\gamma_p|dp$
$= \frac{1}{2}\int k^2 < \gamma, \vec{n} > |\gamma_p|dp - \frac{1}{2}\int k|\gamma_p|dp + \frac{1}{2}\int k_s < \gamma, \vec{t} > |\gamma_p|dp$
$= \frac{1}{2}\int_\gamma k^2 < \gamma, \vec{n} > ds - \frac{1}{2}\int_\gamma kds + \frac{1}{2}\int_\gamma k_s < \gamma, \vec{t} > ds$
$=$ (Int. by parts on last term) $- \int_\gamma kds$.

Thus $A'(t) < 0 \Rightarrow A$ is strictly decreasing. Moreover, $\int_\gamma kds = \int_0^{2\pi} d\theta = 2\pi$. Therefore, $A(t) = A(0) - 2\pi t$. If we can establish that our solution must exist for all $t \in [0, \frac{A(0)}{2\pi})$, then we will establish that A approaches zero in finite time, namely at time $t = \frac{A(0)}{2\pi}$.

THE BLASCHKE SELECTION THEOREM:

HAUSDORFF DISTANCE:
Let $A, B \subset R^2$. We define the Hausdorff distance between the two sets A and B as $d_H(A, B) = inf\{r > 0 : A \subset B + rD, B \subset A + rD\}$, where D is the unit disc in R^2. We then note that for any fixed $X \subset R^2$, the Hausdorff distance defines a metric space on $M(X)$, the set of all closed subsets of X.

Proposition: $d_H(A,B) = max\{sup_{a\in A} d(a,B), sup_{b\in B} d(b,A)\}$. Here d is the usual Euclidean distance.

Proof: W.L.O.G. we will assume $\lambda = sup_{a\in A} d(a,B) = max\{sup_{a\in A} d(a,B), sup_{b\in B} d(b,A)\}$. Let $d = d_H(A,B)$. Then for any $a \in A$, $a \in B + \lambda D \Rightarrow A \subset B + \lambda D$. Likewise, for any $b \in B$, $b \in A + \lambda D \Rightarrow B \subset A + \lambda D$. Thus, $\lambda \geq d_H(A,B)$.

Fix an arbitrary $\varepsilon > 0$. Then for any $d < \mu \leq d + \varepsilon$, $A \subset B + \mu D$ and $B \subset A + \mu D$. Therefore, for any $a \in A$, $a \in B + \mu D \Rightarrow d(a,B) \leq \mu \Rightarrow sup_{a\in A} d(a,B) \leq \mu$. Thus $\lambda \leq \mu \leq d + \varepsilon$. We now let $\varepsilon \to 0+ \Rightarrow \lambda \leq d$. Q.E.D.

Proposition: Let $\{A_i\}_{i=1}^{\infty} \subset M(X)$, $A_i \to A \in M(X)$ with respect to the Hausdorff metric, then
(1) $A = \{a \in X : \forall n \exists a_n \in A_n; a_n \to a\}$ $(=: B)$
(2) $A = \cap_{n=1}^{\infty} \overline{(\cup_{m=n}^{\infty} A_m)}$

Proof: Suppose $d_H(A_i, A) \to 0$ as $i \to \infty$. Let $a \in A$, we want to find a sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n \to a$ and $a_n \in A_n$ $\forall n$.

By our previous proposition we have that $d(a, A_n) \to 0$ as $n \to \infty$. Therefore, given $\varepsilon > 0$ there exists an N such that $n > N \Rightarrow d(a, A_n) < \frac{\varepsilon}{2}$. Thus we can find an $a_n \in A_n$ such that $d(a, a_n) < \varepsilon$ $\forall n > N$. Thus $\{a_n\}$ is our desired sequence.

Thus we've shown that $A \subset B$.

Now take any $a \in B$, i.e. any $a \in X$ such that there exists a sequence $\{a_n\}$ so that $a_n \in A_n$ and $a_n \to a$ as $n \to \infty$. We want to show $a \in A$.

By the hypothesis of the proposition, $\forall i$ $\exists r_i > 0$ such that $A_i \subset A + r_i D$ and $A \subset A_i + r_i D$ with $r_i \to 0$ as $i \to \infty$. $a_n \to a \Rightarrow$ Given $\varepsilon > 0$ $\exists N$ such that $n > N \Rightarrow d(a_n, a) < \varepsilon$. Thus $a \in A_n + \varepsilon D$ $\forall n > N$ and $A_n \subset A + r_n D$. Therefore $a \in (A + r_n D) + \varepsilon D \Rightarrow a \in A + 2(r_n + \varepsilon) D$ $\forall n > N$. Let $n \to \infty$ to get $a \in A + 2\varepsilon D$. We now let $\varepsilon \to 0+$ to get $a \in \bar{A} \Rightarrow a \in A$. Thus we've established (1) in the proposition.

We now define $B := \cap_{n=1}^{\infty}\overline{(\cup_{m=n}^{\infty}A_m)}$, and let $a \in B$. Thus for any n $a \in \overline{(\cup_{m=n}^{\infty}A_m)}$. Therefore there exists a sequence of points $\{b_i^n\}_{i=1}^{\infty}$ such that $b_i^n \to a$ as $i \to \infty$ and $b_i^n \in \cup_{m=n}^{\infty}A_m$ $\forall i$ and thus for each i, $b_i^n \in A_{m_i}$ for some $m_i \geq n$. Thus by using a diagonalization argument, we can find a sequence $\{a_k\}_{k=1}^{\infty}$ such that $a_k \to a$, $a_k \in A_{n_k}$ with $\{n_k\}_{k=1}^{\infty}$ a strictly increasing sequence of integers. But then it follows that $a \in A$ since A is also the limit of the sequence $\{A_{n_k}\}_{k=1}^{\infty}$ with respect to the Hausdorff metric.

Now suppose that $a \in A$, we want to show that $a \in B$. We know there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \to a$ and $a_n \in A_n$ $\forall n$. Therefore $d(a_n, a) = r_n \to 0$ as $n \to \infty$. Thus $a \in \overline{(\cup_{m=n}^{\infty}A_m)}$ $\forall n$. Thus we have (2).
Q.E.D.

Proposition: Suppose $A_i \to A$ in $M(R^n)$ with A_i convex, then A is convex.

Proof: Let $a, b \in A$, then there exists sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $a_n \to a$ and $b_n \to b$, with $a_n, b_n \in A_n$ $\forall n$. For any $t \in [0,1]$, let $c_n := ta_n + (1-t)b_n$. Then A_n convex $\Rightarrow c_n \in A_n$. Also, $c_n \to ta + (1-t)b$ as $n \to \infty \Rightarrow ta + (1-t)b \in A \Rightarrow A$ is convex.
Q.E.D.

Theorem: X is complete $\Rightarrow M(X)$ is complete.

Proof: Suppose that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $M(X)$. Let $S := \{x \in X :$ for any neighborhood U of x, $S_n \cap U \neq \phi$ for infinitely many $n\}$. We'll show $S_n \to S$.

Let $\varepsilon > 0$, and let N be such that $\forall n, m > N$, $d_H(S_n, S_m) < \varepsilon$.

First we'll show that for $x \in S$, $n > N$ we have $d(x, S_n) < 2\varepsilon$:

Let $x \in S$, then $\exists m > N$ such that $B_\varepsilon(x) \cap S_m \neq \phi$, i.e. $\exists y \in S_m$ such that $d(x,y) < \varepsilon$. $d_H(S_m, S_n) < \varepsilon \Rightarrow d(y, S_n) < \varepsilon$. Therefore, $d(x, S_n) \leq d(x,y) + d(y, S_n) < 2\varepsilon$.

Next we'll show that for any $x \in S_n$, $n > N$ we have $d(x, S) < 2\varepsilon$.

We shall define a strictly increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ as follows: Let $n_1 = n$ and for each $k > 1$ let n_k be such that $n_k > n_{k-1}$ and $d_H(S_p, S_q) < \frac{\varepsilon}{2^k}$ $\forall p, q \geq n_k$.

We then define a sequence of points $\{x_k\}_{k=1}^{\infty}$ with $x_k \in S_{n_k}$ as follows: $x_1 = x$ and we choose $x_k \in S_{n_k}$ such that $d(x_{k-1}, x_k) < \frac{\varepsilon}{2^{k-1}}$. We note that this is possible since $d_H(S_{n_{k-1}}, S_{n_k}) < \frac{\varepsilon}{2^{k-1}}$.

Now $\{x_k\}$ is a Cauchy sequence since $\sum_{k=1}^{\infty} d(x_k, x_{k+1}) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = 2\varepsilon < \infty$. Therefore for some $y \in X$, $x_k \to y$. Moreover, $d(x, y) = \lim_{n\to\infty} d(x, x_n)$ $\leq \sum d(x_k, x_{k+1}) \leq 2\varepsilon$. Also, $y \in S$ by construction, therefore $d(x, S) \leq 2\varepsilon$.
Q.E.D.

Theorem: X is compact $\Rightarrow M(X)$ is compact.

Proof: By the previous theorem, $M(X)$ is complete. Therefore, it suffices to show that $M(X)$ is totally bounded.

Recall that for any $\varepsilon > 0$, a set $S \subset X$ is called an ε-net if $d(x, S) \leq \varepsilon$ $\forall x \in X$, and X is totally bounded if there exists a finite ε-net in X.

Let S be a finite ε-net in X.

We'll prove 2^S is an ε-net in $M(X)$. Let $A \in M(X)$. Consider the set $S_A \in 2^S$ defined by $S_A = \{x \in S : d(x, A) \leq \varepsilon\}$. Because S is an ε-net in X, $\forall y \in A$ $\exists x \in S$ such that $d(x, y) \leq \varepsilon$. $d(x, A) \leq d(x, y) \leq \varepsilon \Rightarrow x \in S_A$. Therefore, $d(y, S_A) \leq \varepsilon$ $\forall y \in A$. Also by the definition of S_A, $d(x, A) \leq \varepsilon$ for any $x \in S_A$. Thus, $d_H(A, S_A) \leq \varepsilon \Rightarrow$ (Since A was arbitrary) 2^S is an ε-net in $M(X)$.
Q.E.D.

Thus we can conclude the following result:
For any bounded $X \subset R^2$. If $\{S_n\}_{n=1}^{\infty}$ is a bounded sequence of closed and convex sets contained in X, then since $M(X)$ is compact there is a subsequence $\{S_{n_k}\}_{k=1}^{\infty}$ such that $S_{n_k} \to S \in M(X)$. Moreover, S is convex.

SHRINKING TO A POINT:

We have shown that if a solution to (1) exists for $t \in [0, \frac{A(0)}{2\pi})$, then the area must go go zero. However suppose we can only establish that a solution exists on $[0, \omega)$ with $\omega < \frac{A(0)}{2\pi}$. Then our area $A(t)$ will converge to a positive constant as $t \to \omega$. Note that a priori, this is a possibility, and thus this is a case we must rule out.

Proposition: (Strong Separation Principle) Suppose γ and $\tilde{\gamma}$ are two C^4 solutions to (1). Suppose $\gamma(\cdot, 0)$ encloses $\tilde{\gamma}(\cdot, 0)$, then $\gamma(\cdot, t)$ encloses $\tilde{\gamma}(\cdot, t)$ for positive t.

Proof: We assume $\gamma(\cdot, t)$ and $\tilde{\gamma}(\cdot, t)$ are parameterized by θ, where the unit inward normal is $-(\cos\theta, \sin\theta)$. Let h and $\tilde{h}$ be the support functions, and k, $\tilde{k}$ the curvatures of γ and $\tilde{\gamma}$ respectively. We suppose, without loss of generality, $(0, 0)$ is contained in the set enclosed by $\tilde{\gamma}(\cdot, 0)$. Let $w := h - \tilde{h}$. Then what we have is $w(\cdot, 0) > 0$. Also w satisfies the following parabolic initial value problem:

$$w_\tau = k\tilde{k}\{w_{\theta\theta} + w\}$$
$$w|_{\tau=0} > 0$$

We will show that $w \geq 0$. Suppose $w < 0$ somewhere on $S^1 \times [0, T]$. Let $\lambda > sup|_{S^1 \times [0,T]}\{k\tilde{k}\}$. Then the function $v := e^{-\lambda\tau}w$ has a negative minimum at some point, which we will denote $(\bar{\theta}, \bar{\tau})$, and at this point we have: $v_\tau \leq 0$, $v_\theta = 0$ and $v_{\theta\theta} \geq 0$. Also, $v_\tau = k\tilde{k}v_{\theta\theta} + \{k\tilde{k} - \lambda\}v$. However the left hand side is ≤ 0, and the right hand side is strictly positive if $v < 0$ at $(\bar{\theta}, \bar{\tau})$. Thus we must have $w \geq 0$.
Q.E.D.

Theorem: Let γ be a uniformly convex solution to (1). If γ is a maximal solution defined on $I \times [0, \omega)$ (i.e. we cannot extend our solution to a bigger time interval) and ω is finite, then the area inclosed by $\gamma(\cdot, t) \to 0$ as $t \nearrow \omega$ (and thus $\omega = \frac{A(0)}{2\pi}$ by our representation of $A(t)$ discussed earlier).

Proof: By the previous proposition, all our curves $\gamma(\cdot, t)$ are contained inside of a disk of radius $diam(\gamma_0)$.

We will now suppose that the area enclosed by $\gamma(\cdot,t)$ does not tend to 0 as $t \nearrow \omega$. Let $\{t_j\}_{j=1}^{\infty}$ be any sequence such that $t_j \nearrow \omega$. We have shown before that $A(t_j) = A(0) - 2\pi t_j$, therefore $\lim_{j\to\infty} A(t_j) > 0$ (since we are assuming $\omega < \frac{A(0)}{2\pi}$). Let A_j denote the set which $\gamma(\cdot,t_j)$ encloses along with its boundary. Then $\{A_j\}_{j=1}^{\infty}$ is a collection of closed convex sets, and thus by the Blaschke Selection Theorem there exists a subsequence $\{A_{j_k}\}_{k=1}^{\infty}$ such that $A_{j_k} \to A$, a convex set. Also, by our assumption above, the area of A is positive. Therefore A contains a disc $D_{4\rho}(x_0,y_0)$.

Next we apply the Strong Separation Principle to $\gamma(\cdot,t)$ and the flow starting at $\partial D_{4\rho}(x_0,y_0)$ to get $D_{2\rho}(x_0,y_0)$ is enclosed by $\gamma(\cdot,t)$ $\forall t$ sufficiently close to ω.

Using (x_0,y_0) as the origin, we introduce the support function $h(\theta,t)$ of $\gamma(\cdot,t)$. Let t_0 be close to ω. Then $h(\theta,t) \geq 2\rho$, since $h(\theta,t) = d((x_0,y_0),l)$, where l is the tangent line passing through the point on the curve $\gamma(\cdot,t)$ whose normal is $-(\cos\theta,\sin\theta)$.

We'll now show that k is uniformly bounded in $[t_0,\omega)$.

We now consider the function $\Phi := \frac{-h_\tau}{h-\rho} = \frac{k}{h-\rho}$, which is greater than 0 since $k > 0$ and $h - \rho \geq \rho > 0$.

Let $T < \omega$ and let $\Phi(\theta_0,\tau_1) = max\{\Phi(\theta,\tau) : (\theta,\tau) \in S^1 \times [t_0,T]\}$. Thus we have at (θ_0,τ_1):
(i) $0 = \Phi_\theta = \frac{-h_{\tau\theta}}{h-\rho} + \frac{h_\tau h_\theta}{(h-\rho)^2}$
(ii) $0 \leq \Phi_\tau = \frac{-h_{\tau\tau}}{h-\rho} + \frac{h_\tau^2}{(h-\rho)^2}$
(iii) $0 \geq \Phi_{\theta\theta} = \frac{-h_{\theta\theta\tau}}{h-\rho} + \frac{2h_{\theta\tau}h_\theta}{(h-\rho)^2} + \frac{-2h_\tau h_\theta^2}{(h-\rho)^3} + \frac{h_\tau h_{\theta\theta}}{(h-\rho)^2}$

Using this we get:
(i) and (iii) $\Rightarrow 0 \geq -h_{\tau\theta\theta} + \frac{h_{\theta\theta}h_\tau}{h-\rho}$ $(*)$
(ii) $\Rightarrow 0 \leq -h_{\tau\tau} + \frac{h_\tau^2}{h-\rho}$ $(**)$
Also, $-h_{\tau\tau} = k_\tau = k^2 k_{\theta\theta} + k^3$, $h_\tau = -k$ and $h_{\theta\theta} + h = \frac{1}{k}$. $(***)$
Therefore using $(**)$ and $(***)$, $0 \leq k^2 k_{\theta\theta} + k^3 + \frac{k^2}{h-\rho} = -k^2 h_{\tau\theta\theta} + k^3 + \frac{k^2}{h-\rho}$ $\leq$ (by $(*)$ and $(***)$) $\frac{k^3 h_{\theta\theta}}{h-\rho} + k^3 + \frac{k^2}{h-\rho} \Rightarrow$ (using this and $h_\tau = \frac{-1}{h_{\theta\theta}+h} = -k$)

$0 \leq \frac{k(h_{\theta\theta}+h-\rho)}{h-\rho} + \frac{1}{h-\rho} = \frac{2-\rho k}{h-\rho} \Rightarrow \Phi = \frac{k}{h-\rho} \leq \frac{2}{\rho(h-\rho)}$.
Thus Φ is uniformly bounded and hence so is k.
Q.E.D.

Theorem:(Shrinking to a Point) Suppose $\gamma(\cdot, t)$ is a uniformly convex solution of (1). Suppose ω is finite and the enclosed area tend to 0 as $t \nearrow \omega$, then $\gamma(\cdot, t)$ shrinks to a point.

Proof: We know that $m(\tau) > 0$ for τ small. Moreover, assuming that $m(\tau) > 0$, we get $liminf_{h\to 0}\frac{m(\tau+h)-m(\tau)}{h} \geq (m(\tau))^3 > 0$. Thus, for h sufficiently small, $\frac{m(\tau+h)-m(\tau)}{h} \geq \frac{1}{2}(m(\tau))^3 \Rightarrow m(\tau+h) \geq \frac{h}{2}(m(\tau))^3 + m(\tau)$. Thus $m(\tau)$ is a non-decreasing function of time and thus we can only have the enclosed area go to zero by shrinking to a point.
Q.E.D.

Thus we have $\omega = \frac{A(0)}{2\pi}$ for the Curve Shortening Problem (1). Thus the above theorems apply to this problem with this value of ω.

LONG TERM BEHAVIOR OF THE FLOW:

Let P denote the point such that $\gamma \to P$ as $t \nearrow \omega = \frac{A(0)}{2\pi}$. By otherwise considering $\gamma(\cdot, t) - P$, we may assume P is the origin. We now consider $\tilde{\gamma}(\cdot, t) = \sqrt{\frac{\pi}{A(t)}}\gamma(\cdot, t)$, where $A(t) = A(0) - 2\pi t$. Note that the area enclosed by $\tilde{\gamma}(\cdot, t)$ is equal to π. In the following discussion, the role of τ has changed. Here $\frac{\partial}{\partial t}$ represents the partial derivative with respect to time holding θ fixed. Let $\tau = \tau(t) = \log[A(t)^{-\frac{1}{2}}] = -\frac{1}{2}\log(A(0) - 2\pi t)$, $\tau_0 = \tau(0)$ $= -\frac{1}{2}\log A(0)$. Thus, $t = \frac{A(0)}{2\pi} - \frac{e^{-2\tau}}{2\pi}$. Looking at τ as a function of t, as $t \to \omega = \frac{A(0)}{2\pi}$ we have $\tau \to \infty$. Therefore $\tilde{\gamma}(\cdot, \tau)$ is defined in $[\tau_0, \infty)$.

Now we wish to examine what the corresponding evolution equations for $\tilde{\gamma}$ are. If $\vec{n}$ is the normal to $\gamma(\cdot, t)$, then since $\vec{n} = \frac{(-\gamma_p^2, \gamma_p^1)}{|\gamma_p|} = \frac{(-\tilde{\gamma}_p^2, \tilde{\gamma}_p^1)}{|\tilde{\gamma}_p|}$, we have $\vec{n}$ is also the normal for $\tilde{\gamma}(\cdot, t)$. In the following discussion we let $\tilde{h}$ be the support function of $\tilde{\gamma}$ and $\tilde{k}$ the curvature of $\tilde{\gamma}$.

The relationship between k,h and $\tilde{k}$,$\tilde{h}$:
$h = <\gamma, (\cos\theta, \sin\theta)> \Rightarrow \tilde{h} = <\sqrt{\frac{\pi}{A(t)}}\gamma, (\cos\theta, \sin\theta)> =$
$\sqrt{\frac{\pi}{A(t)}} <\gamma, (\cos\theta, \sin\theta)> = \sqrt{\frac{\pi}{A(t)}}h.$

Likewise, $\tilde{\gamma}_p = \sqrt{\frac{\pi}{A(t)}}\gamma_p$ and $\tilde{\gamma}_{pp} = \sqrt{\frac{\pi}{A(t)}}\gamma_{pp}$. This implies that $\tilde{k} = \sqrt{\frac{A(t)}{\pi}}k$.

Now we compute the evolution equation of $\tilde{h}$:

$\tilde{h}_\tau = \tilde{h}_t \frac{dt}{d\tau} = \tilde{h}_t \frac{e^{-2\tau}}{\pi} = \tilde{h}_t \frac{A(t)}{\pi}$.
Also, $\tilde{h}_t = \frac{\partial}{\partial t}\{\sqrt{\frac{\pi}{A(t)}}h\} = \sqrt{\frac{\pi}{A(t)}}h_t + (\frac{\pi}{A(t)})^{\frac{3}{2}}h = \sqrt{\frac{\pi}{A(t)}}(-k) + (\frac{\pi}{A(t)})^{\frac{3}{2}}h$.
Therefore, $\tilde{h}_\tau = \sqrt{\frac{A(t)}{\pi}}(-k) + \sqrt{\frac{\pi}{A(t)}}h = -\tilde{k} + \tilde{h}$.

Likewise, the evolution equation of $\tilde{k}$ is given by:

$\tilde{k}_\tau = \tilde{k}_t \frac{dt}{d\tau} = \tilde{k}_t \frac{A(t)}{\pi} = \frac{\partial}{\partial t}\{\sqrt{\frac{A(t)}{\pi}}k\}\frac{A(t)}{\pi} = \frac{A(t)}{\pi}\{-\sqrt{\frac{\pi}{A(t)}}k + \sqrt{\frac{A(t)}{\pi}}k_t\} =$
$-\sqrt{\frac{A(t)}{\pi}}k + (\frac{A(t)}{\pi})^{\frac{3}{2}}k_t$. Also, $k_t = k^2 k_{\theta\theta} + k^3$ and $\tilde{k}_{\theta\theta} = \sqrt{\frac{A(t)}{\pi}}k_{\theta\theta}$.
Therefore, $\tilde{k}_\tau = -\sqrt{\frac{A(t)}{\pi}}k + (\frac{A(t)}{\pi})^{\frac{3}{2}}[k^2 k_{\theta\theta} + k^3] = \tilde{k}^2\tilde{k}_{\theta\theta} + \tilde{k}^3 - \tilde{k}$.

Lemma: $sup_{[0,2\pi]\times[\tau_0,\tau]}(\tilde{k}_\theta^2 + \tilde{k}^2) \leq$
$max\{sup_{[0,2\pi]\times[\tau_0,\tau]}\tilde{k}^2, sup_{[0,2\pi]\times\{\tau_0\}}(\tilde{k}_\theta^2 + \tilde{k}^2)\}$.

Proof: Let $f := \tilde{k}_\theta^2 + \tilde{k}^2$. Suppose $\exists\theta_0, \tau_1 > \tau_0$ such that $f(\theta_0, \tau_1) =$
$sup_{[0,2\pi]\times[\tau_0,\tau_1]}(\tilde{k}_\theta^2 + \tilde{k}^2)$. We claim that $\tilde{k}_\theta(\theta_0, \tau_1) = 0$.

At (θ_0, τ_1):

$0 = f_\theta = 2\tilde{k}_\theta\tilde{k}_{\theta\theta} + 2\tilde{k}\tilde{k}_\theta = 2\tilde{k}_\theta(\tilde{k}_{\theta\theta} + \tilde{k})$.

If $\tilde{k}_\theta \neq 0$, then $\tilde{k}_{\theta\theta} + \tilde{k} = 0$. We want to show this is not possible.

At (θ_0, τ_1) we have:

$0 \leq f_\tau = 2\tilde{k}_\theta\tilde{k}_{\theta\tau} + 2\tilde{k}\tilde{k}_\tau \Rightarrow 0 \leq \tilde{k}_\theta\tilde{k}_{\theta\tau} + \tilde{k}\tilde{k}_\tau$ $(*)$.

$0 \geq f_{\theta\theta} = 2\tilde{k}_{\theta\theta}(\tilde{k}_{\theta\theta} + \tilde{k}) + 2\tilde{k}_\theta(\tilde{k}_{\theta\theta\theta} + \tilde{k}_\theta) = 2\tilde{k}_\theta(\tilde{k}_{\theta\theta\theta} + \tilde{k}_\theta)$.

But then, $\tilde{k}_\tau = \tilde{k}^2(\tilde{k}_{\theta\theta} + \tilde{k}) - \tilde{k} = -\tilde{k}$ at (θ_0, τ_1).

Likewise, $\tilde{k}_{\theta\tau} = 2\tilde{k}\tilde{k}_\theta(\tilde{k}_{\theta\theta} + \tilde{k}) + \tilde{k}^2(\tilde{k}_{\theta\theta\theta} + \tilde{k}_\theta) - \tilde{k}_\theta = \tilde{k}^2(\tilde{k}_{\theta\theta\theta} + \tilde{k}_\theta) - \tilde{k}_\theta$ at (θ_0, τ_1).

Thus $(*)$ becomes: $0 \leq -\tilde{k}^2 + \tilde{k}_\theta\tilde{k}^2(\tilde{k}_{\theta\theta\theta} + \tilde{k}_\theta) - \tilde{k}_\theta^2$ $(**)$.

But, $0 \geq f_{\theta\theta} = 2\tilde{k}_\theta(\tilde{k}_{\theta\theta\theta} + \tilde{k}_\theta)$ at $(\theta_0, \tau_1) \Rightarrow 0 \geq \tilde{k}^2\tilde{k}_\theta(\tilde{k}_{\theta\theta\theta} + \tilde{k}_\theta)$. Therefore, using this and $(**)$ we get $0 \leq -\tilde{k}^2 - \tilde{k}_\theta^2 < 0$, which is a contradiction.

Therefore, $\tilde{k}_\theta(\theta_0, \tau_1) = 0$ and hence the lemma follows.
Q.E.D.

Let $K(\tau) = sup_{\theta\in[0,2\pi]}\tilde{k}(\theta, \tau)$. Then $K(\tau) = \tilde{k}(\theta(\tau), \tau)$ for some $\theta(\tau) \in [0, 2\pi]$. Let T be fixed. As a function of τ, suppose $K(s) = sup_{[\tau_0,T]}K$. Then clearly $K(s) = sup_{[\tau_0,s]}K = sup_{[0,2\pi]\times[\tau_0,s]}\tilde{k}$.

Thus, by the Mean Value Theorem, $\frac{\tilde{k}(\theta(s),s)-\tilde{k}(\theta,s)}{|\theta(s)-\theta|} = |\tilde{k}_\theta(a, s)| \leq sup_{\theta\in[0,2\pi]}|\tilde{k}_\theta(\theta, s)|$, where a is strictly between $\theta(s)$ and θ. Thus, $K(s) - \tilde{k}(\theta, s) \leq |\theta(s) - \theta| sup_{\theta\in[0,2\pi]}|\tilde{k}_\theta(\theta, s)|$.

Now we estimate $sup_{\theta\in[0,2\pi]}|\tilde{k}_\theta(\theta, s)|$:

$$\begin{aligned}
&sup_{\theta\in[0,2\pi]}|\tilde{k}_\theta(\theta, s)| \\
&\leq sup_{[0,2\pi]\times[\tau_0,s]}|\tilde{k}_\theta| \\
&= sup_{[0,2\pi]\times[\tau_0,s]}\sqrt{\tilde{k}_\theta^2} \\
&= \sqrt{sup_{[0,2\pi]\times[\tau_0,s]}\tilde{k}_\theta^2} \\
&\leq \sqrt{sup_{[0,2\pi]\times[\tau_0,s]}(\tilde{k}_\theta^2 + \tilde{k}^2)} \\
&\leq \text{(by the previous lemma)} \sqrt{max\{sup_{[0,2\pi]\times[\tau_0,s]}\tilde{k}^2, sup_{[0,2\pi]\times\{\tau_0\}}(\tilde{k}_\theta^2 + \tilde{k}^2)\}} \\
&= \sqrt{max\{K^2(s), sup_{[0,2\pi]\times\{\tau_0\}}(\tilde{k}_\theta^2 + \tilde{k}^2)\}} \\
&\leq \sqrt{max\{K^2(s), sup_{[0,2\pi]\times\{\tau_0\}}\tilde{k}^2 + sup_{[0,2\pi]\times\{\tau_0\}}\tilde{k}_\theta^2\}} \\
&\leq \sqrt{max\{K^2(s), K^2(s) + sup_{\theta\in[0,2\pi]}\tilde{k}_\theta^2(\theta, \tau_0)\}} \\
&= \sqrt{K^2(s) + sup_{\theta\in[0,2\pi]}\tilde{k}_\theta^2(\theta, \tau_0)}
\end{aligned}$$

$= \sqrt{K^2(s) + C^2}$,
$\leq \sqrt{2}(|K(s)| + C)$
$= \sqrt{2}(K(s) + C)$,
where $C = sup_{\theta\in[0,2\pi]}|\tilde{k}_\theta(\theta, \tau_0)|$.

Therefore, $K(s) - \tilde{k}(\theta, s) \leq \sqrt{2}|\theta(s) - \theta|\{K(s) + C\}$. Assuming that $|\theta(s) - \theta| \leq \frac{1}{2\sqrt{2}}$, we then get $K(s) \leq 2\tilde{k}(\theta, s) + C$.

Now we define $E(\gamma) = \frac{1}{2\pi}\int_\gamma k \log k ds$ for γ a uniformly convex curve. We call E the entropy of γ.

Claim: $E(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \log k d\theta$.

To see this, recall that $h_{\theta\theta} + h = \frac{ds}{d\theta} = \frac{1}{k}$. Therefore, $kds = d\theta \Rightarrow$ $\frac{1}{2\pi}\int \log k d\theta = \frac{1}{2\pi}\int k \log k ds$.

For the following discussion, we will sometimes use the notation
$E(\tau) := E(\tilde{\gamma}(\cdot, \tau))$

Lemma: $\frac{d}{d\tau}E(\tilde{\gamma}(\cdot, \tau)) \leq 0$.

Proof:
$\frac{d}{d\tau}E(\tilde{\gamma}(\cdot, \tau))$
$= \frac{1}{2\pi}\frac{d}{d\tau}\int \log \tilde{k} d\theta$
$= \frac{1}{2\pi}\int \frac{\partial}{\partial\tau} \log \tilde{k} d\theta$
$= \frac{1}{2\pi}\int \frac{\tilde{k}_\tau}{\tilde{k}} d\theta$
$= \frac{1}{2\pi}\int \tilde{k}\tilde{k}_{\theta\theta} + \tilde{k}^2 - 1 d\theta$
$= \frac{1}{2\pi}\int \tilde{k}^2 - 1 - \tilde{k}_\theta^2 d\theta$,
with the last equality from integration by parts.

Therefore, $\frac{d^2E}{d\tau^2}$
$= \frac{1}{2\pi}\int \frac{\partial}{\partial\tau}\{\tilde{k}^2 - \tilde{k}_\theta^2 - 1\}d\theta$
$= \frac{1}{2\pi}\int 2\tilde{k}\tilde{k}_\tau - 2\tilde{k}_\theta\tilde{k}_{\tau\theta} d\theta$
$=$ (by integration by parts) $\frac{1}{2\pi}\int 2\tilde{k}\tilde{k}_\tau + 2\tilde{k}_{\theta\theta}\tilde{k}_\tau d\theta$
$= \frac{1}{2\pi}\int 2\tilde{k}_\tau\{\tilde{k} + \tilde{k}_{\theta\theta}\}d\theta$

$= \frac{1}{\pi} \int \tilde{k}_\tau \{\tilde{k} + \tilde{k}_{\theta\theta} - \frac{1}{\tilde{k}} + \frac{1}{\tilde{k}}\} d\theta$
$= \frac{1}{\pi} \int \tilde{k}_\tau \{\tilde{k} + \tilde{k}_{\theta\theta} - \frac{1}{\tilde{k}}\} d\theta + \frac{1}{\pi} \int \frac{\tilde{k}_\tau}{\tilde{k}} d\theta$
$= \frac{1}{\pi} \int \frac{\tilde{k}_\tau}{\tilde{k}^2} \{\tilde{k}^2 \tilde{k}_{\theta\theta} + \tilde{k}^3 - \tilde{k}\} d\theta + \frac{1}{\pi} \int \frac{\tilde{k}_\tau}{\tilde{k}} d\theta$
$= \frac{1}{\pi} \int \frac{\tilde{k}_\tau^2}{\tilde{k}^2} d\theta + \frac{1}{\pi} \int \frac{\tilde{k}_\tau}{\tilde{k}} d\theta$
$\geq$ (by Hölder's inequality) $2(\frac{1}{2\pi} \int \frac{\tilde{k}_\tau}{\tilde{k}} d\theta)^2 + \frac{1}{\pi} \int \frac{\tilde{k}_\tau}{\tilde{k}} d\theta$
$= 2(E'(\tau))^2 + 2E'(\tau)$.

Therefore, $E''(\tau) \geq 2E'(\tau)(E'(\tau) + 1)$.

Suppose we can pick a τ such that $E'(\tau) > 0$. Because of the inequality we just established, $E'' > 0$ at $\tau \Rightarrow E'$ is an increasing function at τ. Therefore on $[\tau, \infty)$ E' is positive and an increasing function.

By the Mean Value Theorem, $\frac{E'(\tau+h)-E'(\tau)}{h} = E''(\xi)$ for some $\tau < \xi < \tau + h$.
But, $E''(\xi) \geq 2E'(\xi)(E'(\xi) + 1) \geq 2E'(\tau)(E'(\tau) + 1)$
$\Rightarrow \frac{E'(\tau+h)-E'(\tau)}{h} \geq 2E'(\tau)(E'(\tau) + 1)$
$\Rightarrow E'(\tau + h) \geq E'(\tau) + 2hE'(\tau)(E'(\tau) + 1) = (1 + 2h)E'(\tau) + 2h(E'(\tau))^2$.
Picking $h > max\{0, \frac{1}{2E'(\tau)} - \frac{1}{2}\}$ we get $E'(\tau + h) > 1$. Thus E' is eventually greater than 1.

Next we will establish that $E > 1$ eventually, given that $E' > 1$ eventually. Assuming τ is so that $E'(\tau) > 1$, by the Mean Value Theorem, $\frac{E(\tau+h)-E(\tau)}{h}$ $= E'(\xi) > 1$. Therefore, $E(\tau + h) > E(\tau) + h$. We now pick $h > 0$ sufficiently large so that $E(\tau) + h > 1$.

Next we want to show E becomes unbounded at some finite time.

Assuming τ_1 is so that $E'(\tau_1) > 0$ and $\tau > \tau_1$, then $E'' \geq 2E'(E' + 1)$
$\Rightarrow \frac{E''(\tau)}{E'(\tau)} \geq 2(E'(\tau) + 1)$
$\Rightarrow \int_{\tau_1}^{\tau} \frac{E''(s)}{E'(s)} ds \geq 2 \int_{\tau_1}^{\tau} (E'(s) + 1) ds$
$\Rightarrow \log \frac{E'(\tau)}{E'(\tau_1)} \geq 2\{E(\tau) - E(\tau_1) + \tau - \tau_1\}$
$\Rightarrow E'(\tau) \geq E'(\tau_1) e^{-2\{E(\tau_1)+\tau_1\}} e^{2\{E(\tau)+\tau\}}$.

Integrating both sides of the last inequality we obtain that for $\tau_2 < \tau$ so that $E'(\tau_2), E'(\tau) > 0$ (and of course assuming $\tau_2 > \tau_1$):

$E(\tau) - E(\tau_2)$
$\geq E'(\tau_1)e^{-2\{E(\tau_1)+\tau_1\}} \int_{\tau_2}^{\tau} e^{2\{E(s)+s\}}ds$
$\geq E'(\tau_1)e^{-2\{E(\tau_1)+\tau_1\}}e^{\tau_2} \int_{\tau_2}^{\tau} e^{\{2E(s)+s\}}ds.$

Assume that τ_2 is large enough so that $E'(\tau_1)e^{-2\{E(\tau_1)+\tau_1\}}e^{\tau_2} > 1$. Then for $\tau > \tau_2$, $E(\tau) \geq E(\tau_2) + \int_{\tau_2}^{\tau} e^{\{2E(s)+s\}}ds$.

Equivalently, for τ sufficiently large and $h > 0$ we have:
$E(\tau + h) \geq E(\tau) + \int_{\tau}^{\tau+h} e^{\{2E(s)+s\}}ds$
$\geq E(\tau) + he^{\{2E(\tau)+\tau\}}$.

Next we proceed to show E blows up in finite time.

Assume $\tau > 0$ is large enough so that the above inequality holds and so that $E > 1$. We define a sequence $\{\tau_k\}_{k=0}^{\infty}$ by:
$\tau_0 = \tau$
$\tau_1 = \tau + \frac{1}{2}$
$\cdots$
$\tau_n = \tau_{n-1} + \frac{1}{2^n} = \tau + \sum_{k=1}^{n} \frac{1}{2^k}$
Then $\tau_n \nearrow \tau + 1$ as $n \to \infty$.

Also, $E(\tau_1) = E(\tau + \frac{1}{2})$
$\geq E(\tau) + \frac{1}{2}e^{2E(\tau)+\tau}$
$\geq 1 + \frac{1}{2}e^2 \geq 2.$

Assume now $E(\tau_n) \geq 2n$, we want to show $E(\tau_{n+1}) \geq 2(n+1)$.

$E(\tau_{n+1}) = E(\tau_n + \frac{1}{2^n})$
$\geq 2n + \frac{1}{2^n}e^{2\cdot 2n}$
$\geq 2n + e^{2n}$
$\geq 2n + 2 = 2(n+1).$

Therefore, $E(\tau_n) \geq 2n$. Let $n \to \infty$ to get $\tau_n \to \tau + 1$ and $E(\tau_n) \to \infty$.

Therefore E becomes unbounded in finite time, but this is a contradiction since E exists on $[\tau_0, \infty)$. Thus $E'(\tau) \leq 0$ everywhere.
Q.E.D.

Let γ be a uniformly convex curve. Let P be a point on γ such that the unit normal $\vec{n}$ at P is $-(\cos\theta, \sin\theta)$, i.e. $P = \gamma(\theta)$. We then define $w(\theta) = h(\theta) + h(\theta + \pi)$. We call w the "width of the convex curve γ in the direction of $(\cos\theta, \sin\theta)$."

The Geometric description of $w(\theta)$ is as follows: For a point $P_\theta = \gamma(\theta)$ where γ incloses the origin, $h(\theta) =$ the distance from 0 to l_θ, the tangent line to γ at P_θ. The unit normal to $\gamma(\theta)$ is $-(\cos\theta, \sin\theta)$ and the normal to $\gamma(\theta + \pi)$ is $(\cos\theta, \sin\theta)$. Thus the tangent lines l_θ and $l_{\theta+\pi}$ are parallel. Hence having a positive lower bound for w, say $w > A$, would insure that γ cannot fit between two parallel lines of distance less than A apart.

Note that $\tilde{\gamma}(\cdot, \tau)$ encloses the origin. We now proceed to show that $w(\theta) := \tilde{h}(\theta) + \tilde{h}(\theta + \pi)$ is indeed bounded below by a positive constant.

Lemma: Let γ be a closed uniformly convex curve and h the support function for γ, then there exists C such that $w(\theta) \geq Ce^{-E(\gamma)}$ for any θ.

Proof: Let $\theta_0 \in [0, 2\pi]$ be fixed.
Note: For notational purposes, we extend k, h to be 2π-periodic functions defined on all of R.

Then,
$\int_{\theta_0}^{\theta_0+\pi} \frac{\sin(\theta-\theta_0)}{k(\theta)} d\theta = \int_{\theta_0}^{\theta_0+\pi} \sin(\theta - \theta_0)(h_{\theta\theta}(\theta) + h(\theta)) d\theta$
$=$ (by integration by parts) $\int_{\theta_0}^{\theta_0+\pi} -\cos(\theta - \theta_0) h_\theta(\theta) + \sin(\theta - \theta_0) h(\theta) d\theta + \sin(\theta - \theta_0) h_\theta(\theta)|_{\theta=\theta_0}^{\theta=\theta_0+\pi}$
$=$ (by integration by parts) $(-\cos(\theta - \theta_0) h(\theta) + \sin(\theta - \theta_0) h_\theta(\theta))|_{\theta=\theta_0}^{\theta=\theta_0+\pi}$
$= h(\theta_0 + \pi) + h(\theta_0)$
$= w(\theta_0)$.
Therefore, $\int_{\theta_0}^{\theta_0+\pi} \frac{\sin(\theta-\theta_0)}{k(\theta)} d\theta = w(\theta_0)$.

By Jensen's Inequality (using that $-\log$ is convex),
$\log w(\theta_0) - \log\pi = \log\{\frac{1}{\pi} \int_{\theta_0}^{\theta_0+\pi} \frac{\sin(\theta-\theta_0)}{k(\theta)} d\theta\}$
$\geq \frac{1}{\pi} \int_{\theta_0}^{\theta_0+\pi} \log(\sin(\theta - \theta_0)) d\theta - \frac{1}{\pi} \int_{\theta_0}^{\theta_0+\pi} \log(k(\theta)) d\theta$.

Therefore, $\log(w(\theta_0)) \geq \log\pi + \frac{1}{\pi} \int_0^\pi \log\sin\theta d\theta - \frac{1}{\pi} \int_0^\pi \log k(\theta + \theta_0) d\theta$.

Likewise,
$\int_{\theta_0+\pi}^{\theta_0+2\pi} \frac{-\sin(\theta-\theta_0)}{k(\theta)} d\theta = -\int_{\theta_0+\pi}^{\theta_0+2\pi} \sin(\theta-\theta_0)(h_{\theta\theta}(\theta)+h(\theta))d\theta$
$=$ (by integration by parts) $\int_{\theta_0+\pi}^{\theta_0+2\pi} \cos(\theta-\theta_0)h_\theta(\theta) - \sin(\theta-\theta_0)h(\theta)d\theta -$
$\sin(\theta-\theta_0)h_\theta(\theta)|_{\theta=\theta_0+\pi}^{\theta=\theta_0+2\pi}$
$=$ (by integration by parts)
$(-\sin(\theta-\theta_0)h_\theta(\theta)|_{\theta=\theta_0+\pi}^{\theta=\theta_0+2\pi} + \cos(\theta-\theta_0)h(\theta))|_{\theta=\theta_0+\pi}^{\theta=\theta_0+2\pi}$
$= h(\theta_0) + h(\theta_0+\pi)$
$= w(\theta_0)$.

By Jensen's Inequality (using that $-\log$ is convex),
$\log w(\theta_0) - \log \pi$
$= \log\{\frac{1}{\pi}\int_{\theta_0+\pi}^{\theta_0+2\pi} \frac{-\sin(\theta-\theta_0)}{k(\theta)} d\theta\}$
$\geq \frac{1}{\pi}\int_{\theta_0+\pi}^{\theta_0+2\pi} log(-\sin(\theta-\theta_0))d\theta - \frac{1}{\pi}\int_{\theta_0+\pi}^{\theta_0+2\pi} \log k(\theta)d\theta$.

Therefore,
$\log w(\theta_0) \geq log\pi + \frac{1}{\pi}\int_{\pi}^{2\pi} \log(-\sin\theta)d\theta - \frac{1}{\pi}\int_{\pi}^{2\pi} \log k(\theta+\theta_0)d\theta$
$= \log\pi + \frac{1}{\pi}\int_0^{\pi} \log(\sin\theta)d\theta - \frac{1}{\pi}\int_{\pi}^{2\pi} \log k(\theta+\theta_0)d\theta$.

Hence we obtain that:
$\log w(\theta_0)$
$\geq \log\pi + \frac{1}{\pi}\int_0^{\pi} \log(\sin\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log k(\theta+\theta_0)d\theta$
$= \log\pi + \frac{1}{\pi}\int_0^{\pi} \log(\sin\theta)d\theta - E(\gamma)$.

Therefore, $w(\theta_0) \geq \pi e^{\frac{1}{\pi}\int_0^{\pi} \log\sin\theta d\theta} e^{-E(\gamma)} = Ce^{-E(\gamma)}$.

Now we establish that C is finite. $\int_0^{\pi} \log\sin\theta d\theta$
$= 2\int_0^{\frac{\pi}{2}} \log\sin\theta d\theta$ and $\int_0^{\frac{\pi}{2}} \log\sin\theta d\theta = \lim_{a\to 0}\int_a^{\frac{\pi}{2}} \log\sin\theta d\theta$. Note also that for $\theta \in [0, \frac{\pi}{2}]$ we have $\frac{2}{\pi}\theta \leq \sin\theta \leq \theta$. Thus $\log\frac{2}{\pi}\theta \leq \log\sin\theta \leq \log\theta$.
Moreover, for any positive constant B, $\int_a^{\frac{\pi}{2}} \log B\theta d\theta = \frac{1}{B}\int_{Ba}^{\frac{B\pi}{2}} \log u du =$
$\frac{1}{B}\{\frac{B\pi}{2} - Ba + \frac{B\pi}{2}\log(\frac{B\pi}{2}) - Ba\log(Ba)\}$. Moreover, By L'Hôpital's rule $\lim_{a\to 0} Ba\log Ba = \lim_{a\to 0}(-Ba) = 0$. Thus C is finite.
Q.E.D.

Thus we have established that $E(\tilde{\gamma}(\cdot,\tau))$ is bounded from above, and since $w \geq Ce^{-E(\tilde{\gamma}(\cdot,\tau))}$ we have that w is strictly greater than 0 for any τ. In particular, there exists a constant A independent of τ such that $w > A$. Hence $\tilde{\gamma}(\cdot,\tau)$ cannot by placed between two parallel lines with distance closer than A. Also since $\tilde{\gamma}(\cdot,\tau)$ is convex and the enclosed area is always equal to π, this also says that the diameter of $\tilde{\gamma}(\cdot,\tau)$ must also have a finite upper bound, and thus the length of $\tilde{\gamma}(\cdot,\tau)$ has a finite upper bound which we'll call L.

Lemma: $\tilde{k}$ is bounded.

Proof: Suppose that $\tilde{k}$ is unbounded. Let $K(\tau) = sup_{\theta\in[0,2\pi]}\tilde{k}(\theta,\tau)$. As a function of τ, let τ_n be defined by $K(\tau_n) = max_{\tau\in[\tau_0,\tau_0+n]}K(\tau)$. Then clearly $K(\tau) \leq K(\tau_n)$ for $\tau \leq \tau_n$. Moreover, $K(\tau_n) \to \infty$ as $n \to \infty$ since we are assuming $\tilde{k}$ is unbounded.

Recall that $E(\tau)$ is a decreasing function of τ, and for $\tau = \tau_n$ we have $K(\tau) \leq 2\tilde{k}(\theta,\tau) + C$ for $|\theta - \theta(\tau)| \leq \frac{1}{2\sqrt{2}}$.

Therefore,
$2\pi E(0) \geq 2\pi E(\tau)$
$= \int \log \tilde{k}(\theta,\tau) d\theta$
$\geq \int_{|\theta-\theta(\tau)|\leq\frac{1}{2\sqrt{2}}} \log \tilde{k}(\theta,\tau) d\theta + \int_{\{\theta:\tilde{k}(\theta,\tau)<1\}} \log \tilde{k}(\theta,\tau) d\theta$
$= \int_{|\theta-\theta(\tau)|\leq\frac{1}{2\sqrt{2}}} \log \tilde{k}(\theta,\tau) d\theta + \int_{\{\tilde{k}<1\}} \tilde{k} \log \tilde{k} ds$
$\geq \int_{|\theta-\theta(\tau)|\leq\frac{1}{2\sqrt{2}}} \log\{\frac{1}{2}(K(\tau) - C)\} d\theta + \int_{\{\tilde{k}<1\}} \tilde{k} \log \tilde{k} ds$
$= \frac{1}{\sqrt{2}} \log \frac{1}{2} + \frac{1}{\sqrt{2}} \log(K(\tau) - C) + \int_{\{\tilde{k}<1\}} \tilde{k} \log \tilde{k} ds.$

Now we examine $-\int_{\{\tilde{k}<1\}} \tilde{k} \log \tilde{k} ds = \int_{\{\tilde{k}<1\}} \tilde{k} \log \frac{1}{\tilde{k}} ds$.

Note that $\lim_{x\to 0+} x \log \frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} \log x =$ (by L'Hôpital's rule) $\lim_{x\to\infty} \frac{1}{x} = 0$. Letting $f(x) = \frac{1}{x} \log x$, then on $[1,\infty)$ f has a local max at $x = e$ and thus a global max at $x = e$, with $f(e) = \frac{1}{e}$. Therefore, $-\int_{\{\tilde{k}<1\}} \tilde{k} \log \tilde{k} ds = \int_{\{\tilde{k}<1\}} \tilde{k} \log \frac{1}{\tilde{k}} ds \leq \frac{L}{e}$.

Thus, $2\pi E(0) \geq \frac{1}{\sqrt{2}} \log \frac{1}{2} + \frac{1}{\sqrt{2}} \log(K(\tau) - C) - \frac{L}{e}$. Therefore, $K(\tau_n) = K(\tau) \leq C + 2e^{\sqrt{2}(2\pi E(0) + \frac{L}{e})}$. Letting $n \to \infty$ we get a contradiction. Therefore $\tilde{k}$ is bounded from above in $[\tau_0, \infty)$.

Q.E.D.

Let γ be a closed convex curve enclosing the origin (so that $h > 0$). We define $F(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \log h(\theta) d\theta$.

Lemma: For $\tau \geq \tau_0$ sufficiently large, $\frac{d}{d\tau}F(\tilde{\gamma}(\cdot,\tau)) \leq 0$. Also, $F(\tilde{\gamma}(\cdot,\tau))$ is bounded from below. Hence $F(\tilde{\gamma}(\cdot,\tau))$ is bounded for τ sufficiently large.

Proof: For notational purposes, let $F(\tau) := F(\tilde{\gamma}(\cdot,\tau))$.

$$\begin{aligned}
&\tfrac{d}{d\tau}F(\tau) \\
&= \tfrac{1}{2\pi}\tfrac{d}{d\tau}\textstyle\int_0^{2\pi} \log \tilde{h}(\theta,\tau) d\theta \\
&= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \tfrac{\partial}{\partial\tau} \log \tilde{h}(\theta,\tau) d\theta \\
&= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \tfrac{\tilde{h}_\tau(\theta,\tau)}{\tilde{h}(\theta,\tau)} d\theta \\
&= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \tfrac{-\tilde{k}(\theta,\tau)+\tilde{h}(\theta,\tau)}{\tilde{h}(\theta,\tau)} d\theta
\end{aligned}$$

Thus, $\frac{dF}{d\tau} \leq 0 \Leftrightarrow \int -\tilde{k} + \tilde{h} d\theta \leq 0 \Leftrightarrow \int \tilde{h} d\theta \leq \int \tilde{k} d\theta \Leftrightarrow \int h d\theta \leq \frac{e^{-2\tau}}{\pi} \int k d\theta$. Where $A(t) = e^{-2\tau}$ is the area enclosed by γ; $t \in [0, \frac{A(0)}{2\pi})$ is given by $t(\tau) = \frac{1}{2\pi}(A(0) - e^{-2\tau})$ or equivalently $\tau(t) = -\frac{1}{2}\log(A(0) - 2\pi t)$.

However, $\int h d\theta = \int < \gamma, (\cos\theta, \sin\theta) > d\theta \leq \int |\gamma| d\theta = A(t) = e^{-2\tau}$.

Thus, $\int h d\theta \leq A(t)$. If we show $A(t) \leq \frac{A(t)}{\pi}\int k d\theta$ or $\int k d\theta \geq \pi$ we are done.

But $\int k d\theta = \int k^2 ds$, and by Hölder's inequality $2\pi = \int_0^{2\pi} d\theta = \int k ds \leq \int |k| ds \leq \sqrt{\int k^2 ds}\sqrt{\int ds}$.

Therefore, $\sqrt{\int k^2 ds} \geq \frac{2\pi}{\sqrt{L_\gamma}}$, where $L_\gamma = \int ds$ is the length of γ.

Or equivalently, $\int k^2 ds \geq \frac{4\pi^2}{L_\gamma}$ and we want this $\geq \pi$. But for $\tau \geq \tau_0$ sufficiently large (or t sufficiently close to $\frac{A(0)}{2\pi}$), $L_\gamma < 4\pi$. Thus $\int k^2 ds \geq \pi$ for t sufficiently close to $\frac{A(0)}{2\pi}$ or $\tau > \tau_0$ sufficiently large.

Next, we claim that $F(\tau)$ is uniformly bounded from below.

Let M be such that $\tilde{k} \le M$ (we are guaranteed the existence of a finite M by the previous lemma). Since $\tilde{\gamma}$ encloses 0, $\exists$ a disk of radius $\frac{1}{M}$ which goes through 0 (this is true since the boundary of a disc with radius $\frac{1}{M}$ has curvature M). By rotating our coordinate system, we may assume our circle is $\{(x,y) : (x+\frac{1}{M})^2 + y^2 = \frac{1}{M^2}\} = C$. The support function h_C of C is $\frac{1-\cos\theta}{M}$. Thus, $\tilde{h} \ge h_C$
$\Rightarrow F(\tau) = \frac{1}{2\pi}\int \log \tilde{h} d\theta$
$\ge \frac{1}{2\pi}\int \log h_C d\theta$
$= \frac{1}{2\pi}\int \log \frac{1-\cos\theta}{M} d\theta$
$= -\log M + \frac{1}{2\pi}\int_0^{2\pi} \log(1-\cos\theta) d\theta > -\infty.$
Q.E.D.

Now we define $I(\tau) = I(\tilde{\gamma}(\cdot,\tau)) := \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\tilde{h}^2 - \tilde{h}_\theta^2) d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log \tilde{h} d\theta,$
where we assume that $\tau > \tau_0$ is large enough so that $F'(\tau) \le 0$.

Theorem: $I(\tau)$ is bounded for $\tau > \tau_0$ sufficiently large.

Proof: $I'(\tau) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}_\tau \tilde{h} - \tilde{h}_{\theta\tau}\tilde{h}_\theta d\theta \; - \; F'(\tau)$
$=$ (by integration by parts) $\frac{1}{2\pi}\int_0^{2\pi} \tilde{h}_\tau(\tilde{h} + \tilde{h}_{\theta\theta}) d\theta \; - \; F'(\tau)$
$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\tilde{h}_\tau}{\tilde{k}} - \frac{\tilde{h}_\tau}{\tilde{h}} d\theta$
$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\tilde{h}-\tilde{k}}{\tilde{k}} - \frac{\tilde{h}-\tilde{k}}{\tilde{h}} d\theta$
$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\tilde{h}-\tilde{k})^2}{\tilde{h}\tilde{k}} d\theta$

Therefore, $I(\tau) - I(\tau_1) = \int_{\tau_1}^{\tau} \frac{1}{2\pi}\int_0^{2\pi} \frac{(\tilde{h}-\tilde{k})^2}{\tilde{h}\tilde{k}} d\theta dt,$
where we assume $\tau > \tau_1 > \tau_0$ is large enough such that $F'(\tau_1) \le 0$. Note that this implies that I is a non-decreasing function for $\tau > \tau_1$.

Now we want to show that I is bounded from above and below. From the above discussion we know that for $\tau > \tau_0$ sufficiently large, F is a non-increasing function of τ and $F(\tau) \ge -\log M + \frac{1}{2\pi}\int_0^{2\pi} \log(1-\cos\theta) d\theta$ $> -\infty$. Thus to show that I is bounded, we only need show that $\int \tilde{h}^2 - \tilde{h}_\theta^2 d\theta$ is bounded from above and below. However, $\int \tilde{h}^2 - \tilde{h}_\theta^2 d\theta =$ (by integration by parts) $\int \tilde{h}(\tilde{h} + \tilde{h}_{\theta\theta}) d\theta = \int \frac{\tilde{h}}{\tilde{k}} d\theta \ge 0$. Moreover, $\int \frac{\tilde{h}}{\tilde{k}} d\theta = \int \tilde{h} ds$

$\le sup(\tilde{h}) \int ds \le sup(\tilde{h})L < \infty$. Thus, I is bounded from above and below. Moreover, we get that $I'(\tau) \to 0$ as $\tau \to \infty$.
Q.E.D.

Take any sequence $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \to \infty$. Let A_n denote the bounded component of $R^2 \setminus \tilde{\gamma}(\cdot, \tau_n)$. By the Blaschke Selection Theorem there exists a subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ which converges to a convex set A. We let γ denote ∂A and h its support function. Note also that because we have convergence in the Hausdorff metric, $\tilde{h}(\cdot, \tau_{n_k}) \to h$ uniformly, and since $\tilde{h} > 0$ we have $h \ge 0$.

It was shown that for any τ, $\exists \theta(\tau) \in [0, 2\pi]$ such that $\tilde{h}(\theta, \tau) \ge \frac{1-\cos(\theta-\theta(\tau))}{M}$. Thus $\tilde{h}(\theta, \tau_{n_k}) \ge \frac{1-\cos(\theta-\theta(\tau_{n_k}))}{M}$. $\{\theta(\tau_{n_k})\}_{k=1}^{\infty}$ is a bounded sequence of real numbers, therefore there exists a convergent subsequence. Without loss of generality $\theta(\tau_{n_k}) \to \bar{\theta}$ as $k \to \infty$. Thus $h(\theta) \ge \frac{1-\cos(\theta-\bar{\theta})}{M}$. Therefore, if $h = 0$, the ony possibility is $h(\bar{\theta}) = 0$. Without loss of generality, we will assume $\bar{\theta} = 0$ and thus $\{h > 0\} = (0, 2\pi)$. Let $\phi \in C_0^{\infty}[0, 2\pi]$. For k sufficiently large, we have $\theta(\tau_{n_k})$ a point in the complement of $supp(\phi)$. Since $\tilde{h}(\theta, \tau_{n_k}) \ge \frac{1-\cos(\theta-\theta(\tau_{n_k}))}{M}$ we have $\frac{|\phi(\theta)|}{\tilde{h}(\theta,\tau_{n_k})} \le \frac{M|\phi(\theta)|}{1-\cos(\theta-\theta(\tau_{n_k}))}$ which is clearly finite for any θ bounded away from $\theta(\tau_{n_k})$. Also, $\lim_{\theta \to \theta(\tau_{n_k})} \frac{M\phi(\theta)}{1-\cos(\theta-\theta(\tau_{n_k}))}$ = (by L'Hôpital) $M \lim_{\theta \to \theta(\tau_{n_k})} \frac{\phi'(\theta)}{\sin(\theta-\theta(\tau_{n_k}))}$ = (by L'Hôpital) $M \lim_{\theta \to \theta(\tau_{n_k})} \frac{\phi''(\theta)}{\cos(\theta-\theta(\tau_{n_k}))} = 0$. Therefore, $sup_{[0,2\pi]} \frac{|\phi(\theta)|}{\tilde{h}(\theta,\tau_{n_k})} < \infty$.

Theorem: h as just defined is a weak solution to $h_{\theta\theta} + h = \frac{1}{h}$ on $[0, 2\pi]$.

Proof: Let φ be any smooth test function (i.e. $\varphi \in C_0^{\infty}([0, 2\pi])$). We know that $\int_0^{2\pi} \varphi_{\theta\theta} h + (h - \frac{1}{h})\varphi d\theta =$
$\lim_{k\to\infty} \int_0^{2\pi} \varphi_{\theta\theta}(\theta)\tilde{h}(\theta, \tau_{n_k}) + (\tilde{h}(\theta, \tau_{n_k}) - \frac{1}{\tilde{h}(\theta,\tau_{n_k})})\varphi(\theta)d\theta$.

Also,
$|\int_0^{2\pi} \varphi_{\theta\theta}(\theta)\tilde{h}(\theta, \tau_{n_k}) + (\tilde{h}(\theta, \tau_{n_k}) - \frac{1}{\tilde{h}(\theta,\tau_{n_k})})\varphi(\theta)d\theta|$
= (by integration by parts) $|\int_0^{2\pi} (\tilde{h}_{\theta\theta}(\theta, \tau_{n_k}) + \tilde{h}(\theta, \tau_{n_k}) - \frac{1}{\tilde{h}(\theta,\tau_{n_k})})\varphi(\theta)d\theta|$
$= |\int_0^{2\pi} (\frac{1}{\tilde{k}(\theta,\tau_{n_k})} - \frac{1}{\tilde{h}(\theta,\tau_{n_k})})\varphi(\theta)d\theta|$
$= |\int_0^{2\pi} \frac{1}{\tilde{h}(\theta,\tau_{n_k})}(\frac{\tilde{h}(\theta,\tau_{n_k})}{\tilde{k}(\theta,\tau_{n_k})} - 1)\varphi(\theta)d\theta|$

$\leq sup_{[0,2\pi]} \frac{|\varphi|}{\tilde{h}(\cdot,\tau_{n_k})} | \int_0^{2\pi} (\frac{\tilde{h}(\theta,\tau_{n_k})}{\tilde{k}(\theta,\tau_{n_k})} - 1) d\theta|$
$= sup_{[0,2\pi]} \frac{|\varphi|}{\tilde{h}(\cdot,\tau_{n_k})} | \int_0^{2\pi} \sqrt{\frac{\tilde{k}(\theta,\tau_{n_k})}{\tilde{h}(\theta,\tau_{n_k})}} \cdot \sqrt{\frac{\tilde{h}(\theta,\tau_{n_k})}{\tilde{k}(\theta,\tau_{n_k})}} \cdot (\frac{\tilde{h}(\theta,\tau_{n_k})}{\tilde{k}(\theta,\tau_{n_k})} - 1) d\theta|$
$\leq$ (Hölder's inequality)
$sup_{[0,2\pi]} \frac{|\varphi|}{\tilde{h}(\cdot,\tau_{n_k})} (\int_0^{2\pi} \frac{\tilde{h}(\theta,\tau_{n_k})}{\tilde{k}(\theta,\tau_{n_k})} d\theta)^{\frac{1}{2}} (\int_0^{2\pi} \frac{\tilde{k}(\theta,\tau_{n_k})}{\tilde{h}(\theta,\tau_{n_k})} \cdot (\frac{\tilde{h}(\theta,\tau_{n_k})}{\tilde{k}(\theta,\tau_{n_k})} - 1)^2 d\theta)^{\frac{1}{2}}$.

However, $\int_0^{2\pi} \frac{\tilde{k}(\theta,\tau_{n_k})}{\tilde{h}(\theta,\tau_{n_k})} \cdot (\frac{\tilde{h}(\theta,\tau_{n_k})}{\tilde{k}(\theta,\tau_{n_k})} - 1)^2 d\theta = \int_0^{2\pi} \frac{(\tilde{h}(\theta,\tau_{n_k}) - \tilde{k}(\theta,\tau_{n_k}))^2}{\tilde{h}(\theta,\tau_{n_k})\tilde{k}(\theta,\tau_{n_k})} d\theta =$
$2\pi I'(\tau_{n_k}) \to 0$ as $k \to \infty$. Note that we have already shown that $\int \frac{\tilde{h}}{\tilde{k}} d\theta$ is bounded independent of τ. Thus h is a weak solution to $h_{\theta\theta} + h = \frac{1}{h}$ on $[0, 2\pi]$.
Q.E.D.

We know that $\frac{1}{h} \in L^2_{loc}[0, 2\pi]$ and thus by the Interior Regularity Theorem (see Evans sec 6.3) we get $h \in H^2_{loc}[0, 2\pi]$. By the chain rule for weak derivatives we get that $\frac{1}{h} \in H^2_{loc}[0, 2\pi]$. Thus again applying the Interior Regularity Theorem we get $h \in H^4_{loc}[0, 2\pi]$. Thus by an inductive argument, we get for any positive integer k, $h \in H^k_{loc}[0, 2\pi]$. Therefore, by the General Sobolev Inequality (see Theorem 6 in sec 5.7 of Evans) we get
$h \in C^\infty_{loc}[0, 2\pi]$.

Note that $h_{\theta\theta} = \frac{1-h^2}{h}$. Thus if $h < 1$, then $h_{\theta\theta} > 0$. Therefore h is convex for all θ such that $h < 1$, and thus h_θ is bounded for values of θ such that $h(\theta) < 1$. Fix $\alpha < \beta \in (0, 2\pi)$. Then $h_\theta(\beta) - h_\theta(\alpha) = \int_\alpha^\beta h_{\theta\theta}(\theta) d\theta = \int_\alpha^\beta \frac{1}{h} d\theta$ $- \int_\alpha^\beta h d\theta$. Therefore, $h_\theta(\beta) - h_\theta(\alpha) + \int_\alpha^\beta h d\theta = \int_\alpha^\beta \frac{1}{h} d\theta$. The left hand side is finite for any $\alpha < \beta \in (0, 2\pi)$. Letting $\alpha \to 0$ or $\beta \to 2\pi$ we get
$\int_0^{2\pi} \frac{1}{h} d\theta < \infty$.

At this point we want to establish $h \neq 0$ on $[0, 2\pi]$. We know that $h_{\theta\theta} = \frac{1-h^2}{h}$. Thus, on $(0, 2\pi)$ $h_\theta h_{\theta\theta} + h_\theta h = \frac{h_\theta}{h} \Rightarrow \frac{d}{d\theta}\{h_\theta^2 + h^2\} = \frac{d}{d\theta} \log h \Rightarrow$ $h^2 = \log h - h_\theta^2 + C$. Thus, if h we to get arbitrarily small, then $\log h$ would not be bounded from below. However, as $h \to 0$ we have h_θ stays bounded. Thus if $h \to 0$, then $\log h - h_\theta^2 + C \to -\infty$, which cannot happen. Thus h remains positive, and thus in particular, $\frac{1}{h} \in L^2[0, 2\pi]$. Using this we may establish global regularity. To do so, we refer the reader to Gilbarg and Trudinger chapter 8, in particular section 8.4. Thus we may conclude that $h \in C^\infty[0, 2\pi]$.

Suppose h is a 2π-periodic solution to $(*)$ $h_{\theta\theta} + h = \frac{1}{h}$, we will show that $h \equiv 1$.

Notice first that since $h_{\theta\theta} = \frac{1-h^2}{h}$. At a local maximum $h_{\theta\theta} \leq 0 \Rightarrow h \geq 1$ at a local max. Likewise at a local minimum $h \leq 1$.

Multiplying both sides of $(*)$ by h_θ we get $h_\theta h_{\theta\theta} + hh_\theta = \frac{h_\theta}{h} \Rightarrow \frac{d}{d\theta}[\frac{1}{2}(h_\theta^2 + h^2) - \log h] = 0$. Thus, $\frac{1}{2}(h_\theta^2 + h^2) - \log h = C$. Notice that at a local max/min this reduces to $\frac{1}{2}h^2 - \log h = C$.

Suppose M is the maximum of h on $[0, 2\pi]$ and m is the minimum of h on $[0, 2\pi]$. Then $\frac{1}{2}M^2 - \log M = \frac{1}{2}m^2 - \log m \Rightarrow \frac{1}{2}(M^2 - m^2) = \log \frac{M}{m}$. By examining the function $f(x) = \frac{1}{2}x^2 - \log x$ for $x > 0$ we get that $f'(x) = x - \frac{1}{x} > 0$ for $x > 1$, $f' = 0$ at $x = 1$ and $f' < 0$ for $x < 1$. Also, $f''(x) = 1 + \frac{1}{x^2} > 0$. Thus $f = C$ has 2 unique solutions for $C > \frac{1}{2}$, 1 solution for $C = \frac{1}{2}$ and no solution for $C < \frac{1}{2}$. Thus, given M, m is uniquely defined. Moreover, as $M \to \infty$ we have $m \to 0$ and vice versa.

Let $h(\cdot, M)$ denote the solution to $(*)$ with $sup_{[0,2\pi]}h = M$. Suppose $M > 1$ (thus $m < 1$) and h is 2π-periodic. By translation we may assume $h(0) = M$ and thus $h_\theta(0) = 0$. Then $u = h_\theta$ satisfies $u_{\theta\theta} + (1 + \frac{1}{h^2})u = 0$. But there exists $\bar{\theta} \in (0, 2\pi)$ such that $h(\bar{\theta}) = m \Rightarrow u(\bar{\theta}) = m$ and thus $u_{\theta\theta}(\bar{\theta}) = 0$. Thus we get that h is symmetric with respect to $\bar{\theta} \Rightarrow$ the minimal period T of h is at most π.

Next we will show $(h^2)_{\theta\theta\theta} + 4(h^2)_\theta = \frac{4h_\theta}{h}$. To see this note that $(h^2)_{\theta\theta\theta} = 2hh_{\theta\theta\theta} + 6h_\theta h_{\theta\theta}$, and $4(h^2)_\theta = 4hh_\theta$. Thus making the appropriate substitutions for $h_{\theta\theta\theta}$ and $h_{\theta\theta}$ we get our result.

Therefore,
$4\int_0^T \frac{h_\theta}{h} \cos 2(\theta - \theta_0) d\theta = \int_0^T \cos 2(\theta - \theta_0) \cdot [(h^2)_{\theta\theta\theta} + 4(h^2)_\theta] d\theta =$ (by integration by parts) $\cos 2(\theta - \theta_0) \cdot (h^2)_{\theta\theta}|_0^T + 2 \sin 2(\theta - \theta_0) \cdot (h^2)_\theta|_0^T$. Letting $2\theta_0 = T - \frac{\pi}{2}$ the right hand side becomes $-\sin(2\theta - T) \cdot (h^2)_{\theta\theta}|_0^T + 2\cos(2\theta - T) \cdot (h^2)_\theta|_0^T = 4 \sin T \cdot (h^2(0) - 1)$ which is non-negative since $T \leq \pi$. On the other hand, the left hand side equals $-4\int_0^T \frac{h_\theta}{h} \sin(2\theta - T) d\theta$. But for $\theta \in [0, \frac{T}{2}]$, $h_\theta < 0$ and $\sin(2\theta - T) < 0$, and for $\theta \in [\frac{T}{2}, T]$, $h_\theta > 0$ and $\sin(2\theta - T) > 0$. Therefore, its left-hand side is non-positive. Thus giving us a contradiction (because if it were 0 then

$h_\theta \equiv 0$ and we assumed otherwise). Thus the only 2π-periodic solution to $(*)$ is $h(\cdot, 1) \equiv 1$.

Thus we have $h \equiv 1 \Rightarrow \tilde{\gamma}(\cdot, \tau_{n_k}) \rightarrow$ the unit circle. But then this is true for any convergent subsequence $\Rightarrow \tilde{\gamma} \rightarrow$ the unit circle.

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