Notes on the Yamabe Flow

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Chapter 1

The Yamabe Flow for M Compact without Boundary

1.1 Introduction

Let $M = M^n$ be a connected compact Riemannian manifold without boundary; $n \geq 3$. Let $[g_0]$ be a given conformal class of metrics on M and $g \in [g_0]$. Consider, the flow[1]

$$\frac{\partial g}{\partial t} = \frac{n-2}{2nV(g)}(s_g - R_g)g \tag{1.1}$$

where $s_g = \frac{1}{V(g)} \int_M R_g dV_g$, R_g is the scalar curvature with respect to the metric g, and $V(g) = \int_M dV_g$. Note that a subscript of 0 means quantities with respect to g_0, some fixed metric. This may be the metric $g|_{t=0}$ or some other fixed background metric. Also, we define the constant $c = c(n) := \frac{n-2}{4(n-1)}$.

1.2 Some Basics about Conformal Classes

Suppose we have an n dimensional vector space V with two inner products $<,>$ and $<<,>>$. We will first establish that $<,>= c <<,>>$, $c > 0 \iff <,>$ and $<<,>>$ are conformal. First suppose $<,>= c <<,>>$, then for any $u, v \in V$ we have

$$\frac{<u,v>}{|u||v|} = \frac{c << u,v >>}{\sqrt{c}\|u\|\sqrt{c}\|v\|} = \frac{<< u,v >>}{\|u\|\|v\|}.$$

Next, suppose that $<,>$ and $<<,>>$ are conformally equivalent. Let $\{u_i\}_{i=1}^n$ be an $<,>$ orthonormal basis, then $\{u\}_{i=1}^n$ is an $<<,>>$ orthogonal basis. Let $c_i = \|u_i\|^2$, we want to establish that $c_i = c_j$ for $i \neq j$. First, $< u_i + u_j, u_j >=< u_j, u_j >= 1$ for $i \neq j$. Next,

$$|u_i + u_j|^2 =< u_i + u_j, u_i + u_j >=< u_i, u_i > + < u_j, u_j >= 2.$$

Thus

$$\frac{1}{\sqrt{2}} = \frac{< u_i + u_j, u_j >}{|u_i + u_j||u_j|} = \frac{<< u_i + u_j, u_j >>}{\|u_i + u_j\|\|u_j\|} = \frac{c_j}{\sqrt{c_i + c_j}\sqrt{c_j}}.$$

[1]Note that we first consider this flow, which is the negative gradient flow of $S(g) = \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M R_g dV_g$. One can show that $\mathrm{grad}\{S(g)\} = \frac{n-2}{2nV(g)}(R_g - s_g)g$ in an appropriate sense, and we shall show this in the appendix.

This implies $c_i = c_j$.

Next, we have a theorem which tells us that the Yamabe Flow (1.1) preserves conformal class.

THEOREM 1.2.1 *Suppose* $\frac{\partial}{\partial t} g_{ij} = f(g) g_{ij}$, *then* $g_{ij}(t) = e^{w(t)} g_{ij}(0)$.

Proof: Let $\vec{v}, \vec{u} \in T_pM$ for an arbitrary $p \in M$. Let $\phi(t)$ denote the square of the cosine of the angle between $\vec{u}, \vec{v}$ with respect to the metric $g(t)$, i.e.

$$\phi(t) = \frac{<\vec{u}, \vec{v}>^2_{g(t)}}{\|\vec{u}\|^2_{g(t)} \|\vec{v}\|^2_{g(t)}} = \frac{(\sum_{i,j} u^i v^j g_{ij}(t))(\sum_{k,l} u^k v^l g_{kl}(t))}{(\sum_{i,j} u^i u^j g_{i,j}(t))(\sum_{k,l} v^k v^l g_{k,l}(t))},$$

where here $\vec{u} = u^i \frac{\partial}{\partial x^i}$, $\vec{v} = v^i \frac{\partial}{\partial x^i}$ and $g_{ij} =< \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} >_g$ with respect to some local coordinate system $\{x^i\}$. Then a simple computation shows

$$\begin{aligned} \dot{\phi} &= \frac{2(u^i v^j g_{ij})(u^k v^l \dot{g}_{kl})}{(u^i u^j g_{ij})(v^k v^l g_{kl})} - \frac{(u^i v^j g_{ij})^2}{(u^i u^j g_{ij})^2 (v^i v^j g_{ij})^2} \{(u^i u^j g_{ij})(v^k v^l \dot{g}_{kl}) + (u^i u^j \dot{g}_{ij})(v^k v^l g_{kl})\} \\ &= \frac{2f(u^i u^j g_{ij})^2}{(u^i u^j g_{ij})(v^k v^l g_{kl})} - 2f \frac{(u^i u^j g_{ij})^2}{(u^i u^j g_{ij})^2 (v^i v^j g_{ij})^2} \{(u^i u^j g_{ij})(v^i v^j g_{ij})\} \\ &= 0. \end{aligned}$$

Q.E.D.

1.3 The Yamabe Flow Preserves Volume $\int_M dV_g$

We shall now show that (1.1) preserves volume. Let $g \in [g_0]$, then by the above discussion we have $g = f g_0$ with $f > 0$.

THEOREM 1.3.1 *If g is a solution to (1.1) and* $V(t) = V(g(\cdot, t)) := \int_M dV_{g(\cdot,t)}$, *then* $V'(t) = 0$.

Proof: Let

$$V(t) := V(g(t)) = V(g) = \int_M f^{\frac{n}{2}} dV_0.$$

Then,

$$\begin{aligned} V'(t) &= \int_M \partial_t \{f^{\frac{n}{2}}\} dV_0 \\ &= \frac{n}{2} \int_M f^{\frac{n}{2}-1} f_t dV_0 \\ &= \frac{n}{2} \int_M f^{\frac{n}{2}-1} \frac{n-2}{2nV(g)} (s_g - R_g) f dV_0 \\ &= \frac{n-2}{4V(g)} \int_M (s_g - R_g) f^{\frac{n}{2}} dV_0 \\ &= \frac{n-2}{4V(g)} \int_M s_g - R_g dV_g \end{aligned}$$

$$= \frac{n-2}{4V(g)}\{\int_M s_g dV_g - \int_M R_g dV_g\}$$
$$= \frac{n-2}{4}\{s_g - \frac{\int_M R_g dV_g}{V(g)}\}$$
$$= \frac{n-2}{4}\{s_g - s_g\} = 0$$

Q.E.D.

Therefore, we may re-scale time by a constant factor and instead consider the following flow [2]

$$\frac{\partial g}{\partial t} = (s_g - R_g)g \tag{1.2}$$

1.4 Classification of $[g]$ by the Sign of $\lambda_1(-L_g)$

Definition: $[g_0]$ is said to be *scalar positive*, *scalar negative* or *scalar flat* if $[g_0]$ contains a metric of positive, negative or identically zero scalar curvature respectively.[3]

1.4.1 Scalar Curvature and Conformal Metrics

Definition: Given a manifold M with a compatible metric g, we define the **Conformal Laplacian** L_g by

$$L_g u = \triangle_g u - \frac{n-2}{4(n-1)} R_g u.$$

L_g is conformally invariant in the sense that if $\tilde{g} = u^{\frac{4}{n-2}} g$, then we shall show that

$$L_g(uv) = u^{\frac{n+2}{n-2}} L_{\tilde{g}}(v) \tag{1.3}$$

and

$$R_{\tilde{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} L_g u \tag{1.4}$$

are equivalent. In the following theorem, theorem 1.4.1, we will show (1.4) holds for $\tilde{g} = u^{\frac{4}{n-2}} g$.

First assume that $L_g(uv) = u^{\frac{n+2}{n-2}} L_{\tilde{g}}(v)$ for any C^2 function v. Then taking $v \equiv 1$ we get $L_g(u) = -\frac{n-2}{4(n-1)} u^{\frac{n+2}{n-2}} R_{\tilde{g}} u \Rightarrow R_{\tilde{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} L_g u$.

Next assume that $R_{\tilde{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} L_g u$. Then writing $g = v^{\frac{4}{n-2}} g_0$ we get $\tilde{g} = (uv)^{\frac{4}{n-2}} g_0$. Thus, $R_{\tilde{g}} = u^{-\frac{n+2}{n-2}}(-\frac{1}{c}\triangle_g u + R_g u) = (uv)^{-\frac{n+2}{n-2}}(-\frac{1}{c}\triangle_{g_0}(uv) + R_{g_0}(uv))$. This relationship then gives us $L_g u = v^{-\frac{n+2}{n-2}} L_{g_0}(uv)$.

[2] In the future, we may drop the subscript of g for convenience. Thus (1.2) will read $g_t = (s-R)g$.

[3] We will later show that any given conformal class $[g_0]$ is either scalar positive, negative or flat.

Theorem 1.4.1 *For $\tilde{g} = u^{\frac{4}{n-2}} g$ we have*

$$R_{\tilde{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} L_g u.$$

Proof: The scalar curvature is a trace of the Ricci curvature $R = g^{ij} R_{ij}$, $R_{ij} = R^k_{ikj}$, and in local coordinates we have

$$R^k_{lij} = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \sum_m \{\Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}\},$$

where

$$\Gamma^l_{ij} = \frac{1}{2}\sum_k (\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) g^{kl},$$

where $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$, $\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$ and $g_{kj} =< \partial_k, \partial_j >$.

Let $\tilde{g} = e^f g$. For notational purposes, quantities with respect to $\tilde{g}$ will have a tilde over them.

$$\begin{aligned}
\tilde{\Gamma}^l_{ik} - \Gamma^l_{ik} &= \frac{1}{2}\sum_m (\partial_i(e^f g_{mk}) + \partial_k(e^f g_{mi}) - \partial_m(e^f g_{ik}))e^{-f} g^{ml} - \Gamma^l_{ik} \\
&= \frac{1}{2}\sum_m (g_{mk}\partial_i f + g_{mi}\partial_k f - g_{ik}\partial_m f) g^{ml}.
\end{aligned}$$

Now we compute $\tilde{R}^l_{kij} - R^l_{kij}$ at a point $p \in M$, and in doing so we shall assume that we are working in geodesic normal coordinates at p. i.e., $g_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = 0\ \forall i, j, k$ at p.

Thus at p,

$$\tilde{R}^l_{kij} - R^l_{kij} = \partial_i(\tilde{\Gamma}^l_{jk} - \Gamma^l_{jk}) - \partial_j(\tilde{\Gamma}^l_{ik} - \Gamma^l_{ik}) + \sum_m \{\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik}\}.$$

Therefore, at p we have

$$\begin{aligned}
\tilde{R}_{kj} - R_{kj} &= \sum_i (\tilde{R}^i_{kij} - R^i_{kij}) \\
&= \sum_i \{\partial_i(\tilde{\Gamma}^i_{jk} - \Gamma^i_{jk}) - \partial_j(\tilde{\Gamma}^i_{ik} - \Gamma^i_{ik}) + \sum_m \{\tilde{\Gamma}^i_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^i_{jm}\tilde{\Gamma}^m_{ik}\}\} \\
&= \sum_i [\frac{1}{2}\partial_i \sum_m (g_{mk}\partial_j f + g_{mj}\partial_k f - g_{jk}\partial_m f) g^{mi} - \frac{1}{2}\partial_j \sum_m (g_{mk}\partial_i f + g_{mi}\partial_k f - g_{ik}\partial_m f) g^{mi} \\
&\quad + \frac{1}{4}\sum_m \{(\sum_l (g_{lm}\partial_i f + g_{li}\partial_m f - g_{im}\partial_l f) g^{li})(\sum_r (g_{rk}\partial_j f + g_{rj}\partial_k f - g_{jk}\partial_r f) g^{rm}) \\
&\quad - (\sum_l (g_{lm}\partial_j f + g_{lj}\partial_m f - g_{jm}\partial_l f) g^{li})(\sum_r (g_{rk}\partial_i f + g_{ri}\partial_k f - g_{ik}\partial_r f) g^{rm})\}]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i [\frac{1}{2}\sum_m \{g_{mk}\partial_i\partial_j f + g_{mj}\partial_i\partial_k f - g_{jk}\partial_i\partial_m f\}g^{mi} \\
&\quad -\frac{1}{2}\sum_m \{g_{mk}\partial_j\partial_i f + g_{mi}\partial_j\partial_k f - g_{ik}\partial_j\partial_m f\}g^{mi} \\
&\quad +\frac{1}{4}\sum_m \{(\sum_l \{g_{lm}\partial_i f + g_{li}\partial_m f - g_{im}\partial_l f\}g^{li})(\sum_r \{g_{rk}\partial_j f + g_{rj}\partial_k f - g_{jk}\partial_r f\}g^{rm}) \\
&\quad -(\sum_l \{g_{lm}\partial_j f + g_{lj}\partial_m f - g_{jm}\partial_l f\}g^{li})(\sum_r \{g_{rk}\partial_i f + g_{ri}\partial_k f - g_{ik}\partial_r f\}g^{rm})\}] \\
&= I + II + III.
\end{aligned}
$$

Here,

$$
\begin{aligned}
I &= \sum_i [\frac{1}{2}\sum_m \{g_{mk}\partial_i\partial_j f + g_{mj}\partial_i\partial_k f - g_{jk}\partial_i\partial_m f\}g^{mi} \\
&\quad -\frac{1}{2}\sum_m \{g_{mk}\partial_j\partial_i f + g_{mi}\partial_j\partial_k f - g_{ik}\partial_j\partial_m f\}g^{mi}],
\end{aligned}
$$

$$
II = \frac{1}{4}\sum_{i,m}(\sum_l \{g_{lm}\partial_i f + g_{li}\partial_m f - g_{im}\partial_l f\}g^{li})(\sum_r \{g_{rk}\partial_j f + g_{rj}\partial_k f - g_{jk}\partial_r f\}g^{rm})
$$

and

$$
III = -\frac{1}{4}\sum_{m,i}(\sum_l \{g_{lm}\partial_j f + g_{lj}\partial_m f - g_{jm}\partial_l f\}g^{li})(\sum_r \{g_{rk}\partial_i f + g_{ri}\partial_k f - g_{ik}\partial_r f\}g^{rm}).
$$

Now,

$$
\begin{aligned}
I &= \frac{1}{2}\sum_{i,m}\{g_{mj}g^{mi}\partial_i\partial_k f - g_{jk}g^{mi}\partial_i\partial_m f - g_{mi}g^{mi}\partial_j\partial_k f + g_{ik}g^{mi}\partial_j\partial_m f\} \\
&= \partial_j\partial_k f - \frac{n}{2}\partial_j\partial_k f - \frac{1}{2}\sum_{i,m} g_{jk}g^{mi}\partial_i\partial_m f \\
&= \partial_j\partial_k f - \frac{n}{2}\partial_j\partial_k f - \frac{1}{2}g_{jk}\triangle f \\
&= -\frac{n-2}{2}\nabla_j\nabla_k f - \frac{1}{2}g_{jk}\triangle f,
\end{aligned}
$$

$$
\begin{aligned}
4\cdot II &= \sum_{m,r}(\partial_m f + n\partial_m f - \partial_m f)(g_{rk}\partial_j f + g_{rj}\partial_k f - g_{jk}\partial_r f)g^{rm} \\
&= \sum_{m,r} n\partial_m f(g_{rk}\partial_j f + g_{rj}\partial_k f - g_{jk}\partial_r f)g^{rm} \\
&= \sum_m \{n\delta_{mk}\partial_m f\partial_j f + n\delta_{jm}\partial_m f\partial_k f\} - \sum_{m,r} n g_{jk}g^{rm}\partial_m f\partial_r f \\
&= 2n\partial_k f\partial_j f - n\sum_{m,r} g_{jk}g^{rm}\partial_m f\partial_r f
\end{aligned}
$$

$$\Rightarrow II = \frac{n}{2}\nabla_j f \nabla_k f - \frac{n}{4} g_{jk} < \nabla f, \nabla f >,$$

$$-4 \cdot III = \sum_{i,l,m,r} \{g_{lm}\partial_j f + g_{lj}\partial_m f - g_{jm}\partial_l f\} g^{li} \{g_{rk}\partial_i f + g_{ri}\partial_k f - g_{ik}\partial_r f\} g^{rm}$$

$$= \sum_{i,m} \{\delta_{mi}\partial_j f + \delta_{ij}\partial_m f - \sum_l g_{jm} g^{li}\partial_l f\}\{\delta_{km}\partial_i f + \delta_{im}\partial_k f - \sum_r g_{ik} g^{rm}\partial_r f\}$$

$$= \sum_{i,m} \delta_{mi}\delta_{km}\partial_j f \partial_i f + \sum_{i,m} \delta_{mi}\delta_{mi}\partial_j f \partial_k f + \sum_{i,m} \delta_{ij}\delta_{km}\partial_m f \partial_i f$$

$$+ \sum_{i,m} \delta_{ij}\delta_{im}\partial_m f \partial_k f - \sum_{i,r} g_{ik} g^{ri}\partial_j f \partial_r f - \sum_{m,r} g_{jk} g^{rm}\partial_m f \partial_r f$$

$$- \sum_{i,l} g_{jk} g^{li}\partial_l f \partial_i f - \sum_{i,l} g_{ji} g^{li}\partial_l f \partial_k f + \sum_{i,l,m,r} g_{jm} g_{ik} g^{li} g^{rm}\partial_l f \partial_r f$$

$$= \partial_j f \partial_k f + n\partial_j f \partial_k f + \partial_k f \partial_j f + \partial_j f \partial_k f - \partial_j f \partial_k f - \sum_{m,r} g_{jk} g^{rm}\partial_m f \partial_r f$$

$$- \sum_{i,l} g_{jk} g^{li}\partial_l f \partial_i f - \partial_j f \partial_k f + \partial_k f \partial j f$$

$$= (n+2)\partial_j f \partial_k f - 2 g_{jk} \sum_{i,l} g^{li}\partial_l f \partial_i f$$

$$\Rightarrow III = -\frac{n+2}{4}\nabla_j f \nabla_k f + \frac{1}{2} g_{jk} < \nabla f, \nabla f > .$$

Putting this all together, we get

$$\tilde{R}_{kj} - R_{kj} = I + II + III$$

$$= -\frac{n-2}{2}\nabla_k \nabla_j f - \frac{1}{2} g_{jk} \triangle f + \frac{n-2}{4}\nabla_k f \nabla_j f - \frac{n-2}{4} g_{jk} \sum_i \nabla^i f \nabla_i f.$$

Multiplying by g^{jk} and summing over j, k we get

$$R_{\tilde{g}} e^f - R_g = -\frac{n-2}{2}\sum_{j,k} g^{jk}\nabla_k \nabla_j f - \frac{n}{2}\triangle f + \frac{n-2}{4}\sum_{j,k} g^{jk}\nabla_k f \nabla_j f - \frac{n-2}{4} n \sum_i \nabla^i f \nabla_i f$$

$$= -(n-1)\triangle f - \frac{(n-2)(n-1)}{4}\sum_i \nabla^i f \nabla_i f.$$

Now, using $e^f = u^{\frac{4}{n-2}}$ for $u > 0$ we get

$$\triangle f = \frac{4}{n-2}\triangle(\log u)$$

$$= \frac{4}{n-2}\sum_{i,j} g^{ij}(\partial_i \partial_j \log u - (\sum_k \partial_k \log u \cdot \Gamma_{ij}^k))$$

$$= \frac{4}{n-2}\{\frac{1}{u}\sum_{i,j} g^{ij}(\partial_i\partial_j u - (\sum_k \partial_k u \cdot \Gamma_{ij}^k)) - \frac{1}{u^2}\sum_{i,j} g^{ij}\partial_i u \partial_j u\}$$

$$= \frac{4}{n-2}\{\frac{1}{u}\triangle u - \frac{1}{u^2}\sum_i \nabla^i u \nabla_i u\|.$$

Therefore, we get

$$-\frac{4(n-1)}{n-2}\triangle u + R_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}}. \tag{1.5}$$

Q.E.D.

1.4.2 The First Eigenvalue of $-L_g$

Note also that for any $\phi \in C^2$ we have $L_{\tilde{g}}(\phi) = u^{-\frac{n+2}{n-2}} L_g(u\phi)$. Thus

$$\frac{\int \varphi L_{\tilde{g}} \varphi dV_{\tilde{g}}}{(\int \varphi^{\frac{2n}{n-2}} dV_{\tilde{g}})^{\frac{n-2}{n}}} = \frac{\int u\varphi L_g(u\varphi) dV_g}{(\int (u\varphi)^{\frac{2n}{n-2}} dV_g)^{\frac{n-2}{n}}}.$$

Proposition 1.4.2

$$\lambda_1(-L_g) = \inf_{u\in H^1(M)\setminus\{0\}} \frac{\int_M |\nabla_g u|^2 + cR_g u^2 dV_g}{\int_M u^2 dV_g} = \inf_{u\in H^1(M)\setminus\{0\}, \|u\|_{L^2}=1} I(u),$$

where

$$I(u) = \int_M |\nabla_g u|^2 + cR_g u^2 dV_g.$$

Proof: Suppose μ is an eigenvalue of $-L_g$ and $w > 0$ an associated eigenfunction. Then $-\triangle_g w + cR_g w = \mu w$ in the weak sense, and hence for $v \in H^1$ we have

$$\int_M < \nabla_g v, \nabla_g w > + cR_g v w dV_g = \mu \int_M v w dV_g,$$

so taking $v = w$ we get

$$\mu = \frac{\int_M |\nabla_g w|^2 + cR_g w^2 dV_g}{\int_M w^2 dV_g}.$$

We now define

$$I = \inf_{u\in H^1(M)\setminus\{0\}} \frac{\int |\nabla u|^2 + cRu^2 dV}{\int u^2 dV},$$

and we wish to show first that $I > -\infty$. If we show that I is also an eigenvalue, then clearly we must have $I = \lambda_1$ by our above observation.

Proposition 1.4.3 *Assume that* $\sup_M |R_g| < \infty$, *then*

$$I = \inf_{u\in H^1(M)\setminus\{0\}} \frac{\int |\nabla u|^2 + cRu^2 dV}{\int u^2 dV} > -\infty.$$

Proof:

$$\begin{aligned} -I &= \sup_{u\in H^1(M,g),\|u\|_{L^2(M,g)}=1} (-I(u)) \\ &\leq \sup_{u\in H^1(M,g),\|u\|_{L^2(M,g)}=1} \int_M -cR_g u^2 dV_g \\ &\leq \sup_{u\in H^1(M,g),\|u\|_{L^2(M,g)}=1} c\sup_M |R_g| \int_M u^2 dV_g \\ &= c\sup_M |R_g| < \infty. \end{aligned}$$

Q.E.D.

Now Take a minimizing sequence $\{u_j\} \subset H^1(M,g)$ with $\|u_j\|_{L^2(M,g)} = 1$ (thus $I(u_j) \to I$). For j sufficiently large,

$$I + 1 \geq I(u_j) \geq \int |\nabla u_j|^2 dV - c\sup |R| \int u_j^2 dV.$$

Thus,

$$\int |\nabla u_j|^2 dV \leq I + 1 + c\sup |R|$$

$\Rightarrow \{u_j\}$ is a bounded sequence in $H^1(M)$.

Therefore, by the compact embedding of $H^1 \to L^2$, $\exists$ a subsequence which converges in L^2 to some limit u. Without loss of generality, we'll assume the subsequence is the sequence $\{u_j\}$ itself.

Moreover, since H^1 is reflexive and $\|u_j\|_{H^1} \leq C$, $\exists$ a subsequence (which we'll assume is the original sequence) which converges weakly to some limit function in H^1. Moreover, it follows that this limit must also be u.

Next we will use the following fact: For a Banach space H, if $x_n \rightharpoonup x$, then $\{x_n\}$ is norm bounded and $\|x\|_H \leq \liminf_{n\to\infty} \|x_n\|_H$. This is true since there is an $l \in H^*$ such that $\|x\|_H = |l(x)|$, $\|l\|_{H^*} = 1$, $l(x) = \lim_{n\to\infty} l(x_n)$ and $|l(x_n)| \leq \|l\|_{H^*}\|x_n\|_H = \|x_n\|_H$.

Therefore,

$$\int |\nabla u|^2 + u^2 dV \leq \liminf_{n\to\infty} \int |\nabla u_n|^2 + u_n^2 dV$$

Now $\int cRu_j^2 dV \to \int cRu^2 dV$. This is true since,

$$|\int cR(u_j^2 - u^2)dV| \leq c\sup |R| \int |u_j^2 - u^2| dV \to 0 \quad \text{as} \quad j \to \infty.$$

Since

$$\int_M |\nabla u_n|^2 + cRu_n^2 dV \to \int |\nabla u|^2 + cRu^2 dV$$

we get

$$|\int_M (|\nabla u_n|^2 - |\nabla u|^2) + cR(u_n^2 - u^2)dV| \to 0.$$

Moreover

$$||\int|\nabla u|^2-|\nabla u|^2dV|-|\int cR(u_n^2-u^2)dV||\leq|\int_M(|\nabla u_n|^2-|\nabla u|^2)+cR(u_n^2-u^2)dV|\to 0.$$

Moreover, since $u_j\to u$ in the L^2 sense, we get

$$\int|\nabla u_n|^2dV\to|\nabla u|^2dV.$$

Therefore, we get that

$$\int|\nabla u|^2+cRu^2dV\leq\lim_{k\to\infty}\int|\nabla u_k|^2+cRu_k^2dV=I$$

$\Rightarrow$

$$\int|\nabla u|^2+cRu^2dV=I,$$

and therefore u is a minimizer.

Now consider $w:=|u|$. By Lemma 7.6 of [5], we get $w\in H^1(M)$. Also, $|\nabla w|^2=|\nabla u|^2$ and $w^2=u^2$. Therefore, $w\geq 0$ is also a minimizer.

Now we define $f(t)=\frac{I(w+tv)}{\|w+tv\|_{L^2}}$ for $v\in H^1$. Then,

$$f'(t)=-2\frac{\int(w+tv)vdV}{(\int(w+tv)^2dV)^2}\int|\nabla(w+tv)|^2+cR(w+tv)^2dV+\frac{2\int<\nabla(w+tv),\nabla v>+cR(w+tv)vdV}{\int(w+tv)^2dV}.$$

Therefore,

$$f'(0)=2\{B(w,v)-I<w,v>_{L^2}\},$$

where $B(u,v)=\int<\nabla u,\nabla v>+cRuvdV$ and $\|w\|_{L^2}=1$.

$$f'(0)=0\quad\Rightarrow\quad B(w,v)-I<w,v>_{L^2}=0\quad\forall v\in H^1.$$

It can be shown that (by standard regularity theorem for elliptic equations) $Lw-Iw=0$, and thus I is an eigenvalue. Moreover, by the strong maximum principle we actually get $w>0$. Q.E.D.

Here we define the **Yamabe invariant** of $(M,[g_0])$ to be

$$Q(M,[g_0])=\inf\{\frac{\int R_gdV_g}{(\int dV_g)^{\frac{n-2}{n}}}:g\in[g_0]\}$$

$$=\inf_{u>0,u\in H^1(M,g_0)}\frac{\int\frac{1}{c}|\nabla_0u|^2+R_0u^2dV_0}{(\int u^{\frac{2n}{n-2}}dV_0)^{\frac{n-2}{n}}},$$

and note that $Q(M,[g_0])$ is a conformal invariant by its definition[4].

[4]Here we point out that the corresponding Euler-Lagrange equation for the functional $J(u):=\int_M\frac{1}{c}|\nabla_0u|^2+R_0u^2dV_0$ subject to the constraint $H(u)=\int_M u^{\frac{2n}{n-2}}dV_0=1$ is

$$-\frac{4(n-1)}{n-2}\Delta_{g_0}u+R_{g_0}u=\frac{n}{n-2}\lambda u^{\frac{n+2}{n-2}}$$

Proposition 1.4.4

$$\lambda_1(M,g_0)=\lambda_1(-L_{g_0})>-\infty$$

implies

$$Q(M,[g_0])>-\infty.$$

Proof: Recall that

$$Q(M,[g_0])=\inf_{u>0,u\in H^1}\frac{\int|\nabla_0 u|^2+cR_0u^2dV_0}{\left(\int u^{\frac{2n}{n-2}}dV_0\right)^{\frac{n-2}{n}}},$$

and

$$\lambda_1=\inf_{u>0,u\in H^1}\frac{\int|\nabla_0 u|^2+cR_0u^2dV_0}{\int u^2dV_0}.$$

Taking any $w>0, w\in H^1(M,g_0)$ we have

$$\frac{\frac{1}{c}\int|\nabla_0 w|^2+cR_0w^2dV_0}{\left(\int w^{\frac{2n}{n-2}}dV_0\right)^{\frac{n-2}{n}}}=\frac{\frac{1}{c}\int|\nabla_0 w|^2+cR_0w^2dV_0}{\int w^2dV_0}\cdot\frac{\int w^2dV_0}{\left(\int w^{\frac{2n}{n-2}}dV_0\right)^{\frac{n-2}{n}}}$$

$$\geq\frac{\lambda_1}{c}\cdot\frac{\int w^2dV_0}{\left(\int w^{\frac{2n}{n-2}}dV_0\right)^{\frac{n-2}{n}}}.$$

Now taking the infimum over all $w>0$ in H^1 we get

$$Q(M,[g_0])\geq\frac{\lambda_1}{c}\inf_{w>0,w\in H^1}\frac{\int w^2dV_0}{\left(\int w^{\frac{2n}{n-2}}dV_0\right)^{\frac{n-2}{n}}}.$$

Noting that $\lambda_1>-\infty$ and

$$0\leq\inf_{w>0,w\in H^1}\frac{\int w^2dV_0}{\left(\int w^{\frac{2n}{n-2}}dV_0\right)^{\frac{n-2}{n}}}\leq V(g_0)^{\frac{2}{n}}<\infty,$$

we get our result.
Q.E.D.

Proposition 1.4.5 *$\lambda_1(-L_g)$ and $Q(M,[g])$ are of the same sign.*

Proof: Hölder's inequality implies

$$\|u^2\|_{L^1(M,g)}\leq(V(g))^{\frac{2}{n}}\|u^2\|_{L^{\frac{n}{n-2}}(M,g)}.$$

Thus,

$$\frac{1}{\|u^2\|_{L^{\frac{n}{n-2}}(M,g)}}\leq\frac{(V(g))^{\frac{2}{n}}}{\|u^2\|_{L^1(M,g)}}.$$

---

where λ is a Lagrange multiplier. Its non-trivial solutions provide us with metrics $g:=u^{\frac{4}{n-2}}g_0$ such that R_g is a constant (namely $R_g=\frac{n}{n-2}\lambda$). Moreover, multiplying the above equation by u and integrating, we get that the sign of λ is determined by the sign of $\int_M\frac{1}{c}|\nabla_0u|^2+R_0u^2dV_0$. Hence, showing that the Yamabe invariant is achieved, resolves the Yamabe problem. Moreover, the sign of Q determines the sign of out metric with constant scalar curvature obtained from the solution to the Euler-Lagrange equation.

If $I(u) > 0$ for all u, then for any $u > 0, u \in H^1$ we have

$$\frac{\frac{1}{c}I(u)}{\|u^2\|_{L^{\frac{n}{n-2}}(M,g)}} \leq \frac{\frac{1}{c}(V(g))^{\frac{2}{n}}I(u)}{\|u^2\|_{L^1(M,g)}}$$

which implies

$$0 \leq Q(M,[g]) \leq \lambda_1(-L_g)\frac{(V(g))^{\frac{2}{n}}}{c}.$$

If $I(u)$ is negative for some $u > 0, u \in H^1$, for such u

$$\frac{\frac{1}{c}I(u)}{\|u^2\|_{L^{\frac{n}{n-2}}(M,g)}} \geq \frac{\frac{1}{c}(V(g))^{\frac{2}{n}}I(u)}{\|u^2\|_{L^1(M,g)}}$$

which implies

$$0 \geq Q(M,[g]) \geq \lambda_1\frac{(V(g))^{\frac{2}{n}}}{c}.$$

Thus we must have either

$$0 \geq Q \geq \lambda_1\frac{V(g)^{\frac{2}{n}}}{c} \tag{1.6}$$

or

$$0 \leq Q \leq \lambda_1\frac{V(g)^{\frac{2}{n}}}{c}. \tag{1.7}$$

Here we note as the fact from the previous proposition:

$$Q \geq \frac{\lambda_1}{c}\inf_{u>0,u\in H^1}\frac{\int u^2 dV_0}{(\int u^{\frac{2n}{n-2}}dV_0)^{\frac{n-2}{n}}}. \tag{1.8}$$

If $\lambda_1 < 0$, then $-L_g u_1 = \lambda_1 u_1$, for some $u_1 > 0$. Thus $I(u_1) = \int \lambda_1 u_1^2 dV_g < 0 \Rightarrow Q < 0$. If $\lambda_1 > 0$, then by (1.8) we have $Q \geq 0$. If $Q < 0$, then (1.6) implies $\lambda_1 < 0$. If $Q > 0$, then (1.7) implies $\lambda_1 > 0$.

Now suppose $Q = 0$ and $\lambda_1 \neq 0$, then if $\lambda_1 < 0$ then we can find a metric with $R < 0 \Rightarrow Q < 0$. Thus the only possibility is that $\lambda_1 > 0$. However, By the Sobolev inequality, we have $\|u\|_{L^{\frac{2n}{n-2}}} \leq C\|\nabla_0 u\|_{L^2}$ for $u \in H_0^1(M, g_0)$ for some constant C. This then implies that $\frac{\|\nabla_0 u\|_{L^2}}{\|u\|_{L^{\frac{2n}{n-2}}}} \geq \frac{1}{C}$. Now since R_{g_0} can be chosen to be positive, we then get Q must be positive (and no less than $\frac{1}{\sqrt{C}}$).

Lastly, if $\lambda_1 = 0$, then clearly (1.6) and (1.7) force $Q = 0$.
Q.E.D.

Now we establish a result we have stated in the introduction, in the following proposition.

Proposition 1.4.6 *If M is a compact manifold, then M admits a compatible metric whose scalar curvature does not change sign. The sign is uniquely determined by the conformal structure, and so there exists three mutually exclusive possibilities: M admits a compatible metric of positive, negative or identically zero scalar curvature.*

Proof: The three possibilities are distinguished by the sign of λ_1, the first (smallest) eigenvalue of the negative of the Conformal Laplacian $L = -L_g$. (Note that $\lambda_1 = \inf_{u>0} \frac{\int |\nabla u|^2 + cRu^2 dV}{\int u^2 dV}$.) Let $u_1 > 0$ denote a positive eigenfunction.

We define $\tilde{g} = u_1^{\frac{4}{n-2}} g$. Then the metric $\tilde{g} = u_1^{\frac{4}{n-2}} g$ has scalar curvature of one sign, since

$$\begin{aligned} R_{\tilde{g}} &= -\frac{4(n-1)}{n-2} u_1^{-\frac{n+2}{n-2}} L_g u_1 \\ &= \frac{4(n-1)}{n-2} u_1^{-\frac{n+2}{n-2}} \lambda_1 u_1 \\ &= \frac{4(n-1)}{n-2} \lambda_1 u^{\frac{4}{n-2}}. \end{aligned}$$

If $\lambda_1 > 0$, then $Q > 0 \Rightarrow$ there doesn't exist a metric $\tilde{g} \in [g]$ such that $R_{\tilde{g}} \leq 0$.

Now suppose $\lambda_1(-L_g) < 0$ and $\exists \tilde{g} \in [g]$ such that $R_{\tilde{g}} \geq 0$. Then clearly, for $v > 0$ in H^1 we have

$$\int -v L_{\tilde{g}} v dV_{\tilde{g}} = \int |\nabla_{\tilde{g}} v|^2 + cR_{\tilde{g}} v^2 dV_{\tilde{g}} \geq 0$$

which implies $Q \geq 0$, which is a contradiction.

If $\lambda_1 = 0$, then $Q = 0$ and $\exists \tilde{g}$ with $R_{\tilde{g}} \equiv 0$. Then take any $\hat{g} \in [g]$, $\hat{g} = u^{\frac{4}{n-2}} \tilde{g}$ and $R_{\hat{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \triangle_{\tilde{g}} u$. If $R_{\hat{g}}$ is of one sign, without loss of generality $R_{\hat{g}} > 0$, then $\triangle_{\tilde{g}} u < 0 \Rightarrow 0 < \int \triangle_{\tilde{g}} u dV_{\tilde{g}} = 0$, which is a contradiction.

Thus the sign of λ_1 determines the sign of $R_{\tilde{g}}$.
Q.E.D.

1.5 A Relationship for $\triangle_g$ and $\triangle_{g_0}$

In local coordinates

$$\triangle_g f = \frac{1}{\sqrt{\det g_{ij}}} \sum_{l,m} \frac{\partial}{\partial x_l} \{ g^{lm} \sqrt{\det g_{ij}} \frac{\partial}{\partial x_m} f \}.$$

Thus, if $g = e^v g_0$, then we have

$$\begin{aligned} \triangle_g f &= e^{-\frac{n}{2}v} \frac{1}{\sqrt{\det(g_0)_{ij}}} \sum_{l,m} \frac{\partial}{\partial x_l} \{ e^{-v} g_0^{lm} \sqrt{\det(g_0)_{ij}} e^{\frac{n}{2}v} \frac{\partial}{\partial x_m} f \} \\ &= e^{-\frac{n}{2}v} \frac{1}{\sqrt{\det(g_0)_{ij}}} \sum_{l,m} \frac{\partial}{\partial x_l} \{ e^{(\frac{n}{2}-1)v} g_0^{lm} \sqrt{\det(g_0)_{ij}} \frac{\partial}{\partial x_m} f \} \end{aligned}$$

$$= e^{-\frac{n}{2}v} \frac{1}{\sqrt{\det(g_0)_{ij}}} \sum_{l,m} \{ e^{(\frac{n}{2}-1)v} \frac{\partial}{\partial x_l} (g_0^{lm} \sqrt{\det(g_0)_{ij}} \frac{\partial}{\partial x_m} f) + g_0^{lm} \sqrt{\det(g_0)_{ij}} \frac{\partial f}{\partial x_m} \frac{\partial v}{\partial x_l} (\frac{n}{2}-1) e^{(\frac{n}{2}-1)v} \}$$

$$\begin{aligned} &= e^{-v} \triangle_{g_0} f + e^{-v} (\frac{n}{2}-1) \sum_{l,m} g_0^{lm} \frac{\partial f}{\partial x_m} \frac{\partial v}{\partial x_l} \\ &= e^{-v} \triangle_{g_0} f + (\frac{n}{2}-1) e^{-v} < \nabla_{g_0} f, \nabla_{g_0} v > . \end{aligned}$$

1.6 Harmonic Functions on Manifolds without Boundary

The Divergence theorem tells us that for any C^2 function f on M we have $\int_M \triangle_g f dV_g = 0$. Since $\frac{1}{2}\triangle_g(f^2) = f\triangle_g f - |\nabla_g f|^2$, we have $\int_M f\triangle_g f - |\nabla_g f|^2 dV_g = 0$. This then gives us the following theorem.

THEOREM 1.6.1 *If $\triangle_g f \geq 0$ (or equivalently, ≤ 0), we get that $\triangle_g f = 0$. Moreover, if $\triangle_g f = 0$, then f is constant.*

Proof: Since $\int_M \triangle_g f dV_g = 0$, we must have $\triangle_g f = 0$ whenever $\triangle_g f \geq 0$ (or ≤ 0). Thus, the above observation then gives us $\int_M |\nabla_g f|^2 dV_g = 0$. Thus $\nabla_g f \equiv 0$, and therefore f is constant on M.
Q.E.D.

THEOREM 1.6.2 *If f is a non-constant function satisfying $\triangle_g f = cf$, for some $c \neq 0$, then $c > 0$.*

Proof: By the above observation, we must have $\int_M cf^2 - |\nabla_g f|^2 dV_g = 0 \Rightarrow c > 0$.
Q.E.D.

Chapter 2

Short Term Existence and Uniqueness of Solutions

2.1 A Parabolic Model for (1.2)

Let $c = c(n) := \frac{n-2}{4(n-1)}$ and $g := u^{\frac{4}{n-2}} g_0 = u^{\frac{4}{n-2}} g_0$. Note that any subscripts which are 0 are subscripts representing quantities with respect to some fixed background metric g_0, which may represent $g|_{t=0}$ or a metric with R_{g_0} of a specific sign. If $L_0 u = \triangle_0 u - cR_0 u$, then $R = R_g = -\frac{1}{c} \frac{L_0 u}{u^{\frac{n+2}{n-2}}}$. We'll show

$$\frac{\partial u^{\frac{n+2}{n-2}}}{\partial t} = \frac{n+2}{n-2}(n-1)\{L_0 u + csu^{\frac{n+2}{n-2}}\} \tag{2.1}$$

We first note that under the assumption $g = u^{\frac{4}{n-2}} g_0$, equation (1.2) reduces to $\frac{\partial}{\partial t} u^{\frac{4}{n-2}} = (s-R)u^{\frac{4}{n-2}}$, or simply $\frac{4}{n-2}\partial_t u = (s-R)u$. However the form stated in (2.1) is more useful for calculations.

From (1.2),

$$\frac{\partial}{\partial t}(u^{\frac{4}{n-2}}) = (s-R)u^{\frac{4}{n-2}},$$

or simply

$$\frac{4}{n-2}\frac{u^{\frac{4}{n-2}}}{u}\partial_t u = (s-R)u^{\frac{4}{n-2}}.$$

Thus,

$$\frac{\partial}{\partial t}(u^{\frac{n+2}{n-2}}) = \frac{n+2}{n-2}u^{\frac{4}{n-2}}\frac{\partial u}{\partial t}$$

$$= \frac{n+2}{4}u\{\frac{4}{n-2}\frac{u^{\frac{4}{n-2}}}{u}\frac{\partial u}{\partial t}\}$$

$$= \frac{n+2}{4}(s-R)u^{\frac{n+2}{n-2}}$$

$$= \frac{n+2}{4}su^{\frac{n+2}{n-2}} + \frac{n+2}{4}u^{\frac{n+2}{n-2}}\{\frac{1}{c}\frac{L_0 u}{u^{\frac{n+2}{n-2}}}\}$$

$$= \frac{n+2}{4} s u^{\frac{n+2}{n-2}} + \frac{(n+2)(n-1)}{n-2} L_0 u,$$

which establishes (2.1).

Note that we may actually study the unnormalized Yamabe flow

$$\frac{\partial}{\partial t} g = -Rg \tag{2.2}$$

Here, the parabolic model for is

$$\frac{\partial}{\partial t}(u^{\frac{n+2}{n-2}}) = \frac{n+2}{4c(n)}\{L_0 u\} \tag{2.3}$$

We will now show that a solution to this simplified problem actually yields a solution to our original problem.

2.2 Short Term Existence for Parabolic Problems

Consider the following Parabolic problem

$$u_t = F(x, t, u, \nabla u, \nabla^2 u) \quad \text{on} \quad M \times (0, \infty),$$

$$u(0) = u_0 \quad \text{on} M, \tag{2.4}$$

with $F, u_0 \in C^\infty$.

The linearization of (2.4) at (t, v) is given by

$$L(t,v)\varphi := \frac{d}{d\varepsilon} F(x, t, v + \varepsilon\varphi, \nabla(v + \varepsilon\varphi), \nabla^2(v + \varepsilon\varphi))|_{\varepsilon=0}.$$

If $L(t, v)$ is elliptic on M, then we say that (2.4) is parabolic at (t, v) . We now prove a local existence result for (2.4).

THEOREM 2.2.1 *Suppose (2.4) is parabolic at $(0, u_0)$, then there exists $\varepsilon > 0$ such that (2.4) has a unique solution $u(x, t)$ for $x \in M$, $t \in [0, \varepsilon]$. Moreover, $u \in C^\infty(M \times [0, \varepsilon])$.*

Proof: First (2.4) is parabolic at (t, u) uniformly with $0 \le t \le T_0$ and $\|u - u_0\|_{C^2(M)} \le \delta_0$, for some $T_0 > 0$, and $\delta_0 > 0$.

Let v be the solution to

$$v_t = L(0, u_0)v + F(x, 0, u_0, \nabla u_0, \nabla^2 u_0) - L(0, u_0)u_0,$$

$$v(0) = u_0. \tag{2.5}$$

We note that

$$\|v(t) - u_0\|_{C^2(M)} \le \delta_0$$

if t is small, say $0 \leq 0 \leq T$.

Define $Q_T := M \times (0, T]$. Let $0 < \alpha < 1$. Define

$$X = \{z \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T) : z(x, 0) = 0\},$$

$$Y = C^{\alpha, \frac{\alpha}{2}}(Q_T).$$

Define $\mathcal{D} : X \to Y$ by $\mathcal{D}(z) = \mathcal{P}(v + z) - \mathcal{P}(v)$,

$$\mathcal{P}(u) = F(x, t, u, \nabla u, \nabla^2 u) - u_t.$$

First $\mathcal{D}(0) = 0$.

$$\mathcal{D}'(0)z = L(t, v)z - z_t, \quad z \in X$$

$\mathcal{D}'(0) : X \to Y$ is invertible because $\mathcal{D}'(0)z = k$, $z|_{t=0} = 0$ is invertible.

By the inverse function theorem, $\exists \delta > 0$ such that if $k \in Y$ with $\|k\|_{Q_T}^{(\alpha, \frac{\alpha}{2})} < \delta$, then there exists a unique function $z \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T)$ satisfying

$$\mathcal{D}(z) = k, \quad z(x, 0) = 0.$$

Let $u = v + z$, $w = \mathcal{P}(v)$, then $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T)$ and

$$\mathcal{P}(u) = w + k \quad \text{on} \quad Q_T,$$

$$u|_{t=0} = u_0.$$

Now we want to show $\mathcal{P}(u) = w + k = 0$ for $0 < t << 1$. Choose a cut-off function $\eta : \mathbb{R} \to \mathbb{R}$, $\eta(t) = 1$ for $t \leq \varepsilon$, $\eta(t) = 0$ for $t \geq 2\varepsilon$, and $0 \leq \eta \leq 1$ for all t. Also $|\eta'(t)| \leq \frac{C}{t}$ for all t. Thus we choose $k = -\eta w$. Now $\|k\|_{Q_T}^{(\alpha, \frac{\alpha}{2})} < \delta$. If ε is small, choose $\alpha < \beta < 1$, then

$$\|\eta w\|_{Q_T}^{(\alpha, \frac{\alpha}{2})} \leq C\varepsilon^{\frac{\beta-\alpha}{2}} \|w\|_{Q_T}^{(\beta, \frac{\beta}{2})}.$$

Thus we get the desired result.
Q.E.D.

2.2.1 Short Term Existence of Solutions to (2.3)

Since (2.3) is a strictly parabolic second order quasi-linear partial differential equation, assuming our initial function u_0 is sufficiently regular, we get that a solution u of (2.3) exists for at least small time t, and is sufficiently regular.

2.3 Solving a Parabolic Problem to Establish Existence of Solutions to (1.2)

Let g be a solution to the initial value problem

$$\frac{\partial}{\partial t}g = -R_g g \tag{2.6}$$

$$g|_{t=0} = g^0, \quad t \in [0, T^*].$$

We want to produce a solution to

$$\frac{\partial}{\partial t}\tilde{g} = (s_{\tilde{g}} - R_{\tilde{g}})\tilde{g}, \tag{2.7}$$

$$\tilde{g}|_{t=0} = g^0, \quad t \in [0, T].$$

Let

$$\tilde{g}(x,t) = \psi(\tau(t))g(x,\tau(t))$$

with $\psi(t) > 0$. Then

$$R_{\tilde{g}}(x,t) = \frac{1}{\psi(\tau(t))}R_g(x,\tau(t)), \quad s_{\tilde{g}}(t) = \frac{1}{\psi(\tau(t))}s_g(\tau(t)).$$

Now,

$$\tilde{g}_t(x,t) = \frac{\partial}{\partial t}\{\psi(\tau(t))g(x,\tau(t))\}$$

$$= \psi'(\tau(t))\tau'(t)g(x,\tau(t)) + \psi(\tau(t))g_t(x,\tau(t))\tau'(t)$$

$$= \psi'(\tau(t))\tau'(t)g(x,\tau(t)) - \psi(\tau(t))R_g(x,\tau(t))g(x,\tau(t))\tau'(t).$$

We need

$$\tilde{g}_t(x,t) = \{s_{\tilde{g}}(t) - R_{\tilde{g}}(x,t)\}\tilde{g}(x,t)$$

$$= \{s_g(\tau(t)) - R_g(x,\tau(t))\}g(x,\tau(t)).$$

Thus we shall require $\psi(\tau(t))\tau'(t) = 1$, so we shall take τ the unique solution to

$$\tau' = \frac{1}{\psi(\tau)}, \quad \tau(0) = 0,$$

that is, $t = \int_0^\tau \psi(\sigma)d\sigma$.

However, we also need

$$s_g(\tau(t)) = \psi'(\tau(t))\tau'(t)$$

$$= \frac{\psi'(\tau(t))}{\psi(\tau(t))}.$$

Thus, we take $\psi(t) = e^{\int_0^t s_g(\sigma)d\sigma}$. Hence, we get a solution $\tilde{g}$ to (2.7), for $t \in [0, T]$ for any T so that $\tau(T) \leq T^*$, using our solution to (2.6).

Likewise, we would like to produce a solution to (2.6) assuming we can solve (2.7). Now, we let $g(x,t) = \varphi(\xi(t))\tilde{g}(x, \xi(t))$ with $\varphi(t) > 0$. Then, $R_g(x,t) = \frac{R_{\tilde{g}}(x,\xi(t))}{\varphi(\xi(t))}$ and $s_g(t) = \frac{s_{\tilde{g}}(\tau(t))}{\varphi(\tau(t))}$.

Now,

$$g_t(x,t) = \frac{\partial}{\partial t}\{\varphi(\xi(t))\tilde{g}(x, \xi(t))\}$$

$$= \varphi'(\xi(t))\xi'(t)\tilde{g}(x, \xi(t)) + \varphi(\xi(t))\tilde{g}_t(x, \xi(t))\xi'(t)$$

$$= \varphi'(\xi(t))\xi'(t)\tilde{g}(x, \xi(t)) + \varphi(\xi(t))\{s_{\tilde{g}}(\xi(t)) - R_{\tilde{g}}(x, \xi(t))\}\tilde{g}(x, \xi(t))\xi'(t).$$

We need

$$g_t(x,t) = -R_g(x,t)g(x,t)$$

$$= -R_{\tilde{g}}(x, \xi(t))\tilde{g}(x, \xi(t)).$$

Thus we shall need $\varphi(\xi(t))\xi'(t) = 1$, so we shall take ξ the unique solution to:

$$\xi' = \frac{1}{\varphi(\xi)}, \quad \xi(0) = 0,$$

that is, $\xi = \xi(t)$ satisfying $t = \int_0^\xi \varphi(\sigma)d\sigma$.

However, we also need:

$$s_{\tilde{g}}(\xi(t)) = -\varphi'(\xi(t))\xi'(t)$$

$$= \frac{-\varphi'(\xi(t))}{\varphi(\xi(t))}.$$

Thus, we take $\varphi(t) = e^{-\int_0^t s_{\tilde{g}}(\sigma)d\sigma}$. Hence, we get a solution g to (2.6), for $t \in [0, T^*]$ for any T^* so that $\xi(T^*) \leq T$, assuming we can solve (2.7).

2.4 Short Term Positivity of Solutions to (2.3)

Assume that u is a C^2 solution to (2.3) with $u_0 > 0$, i.e.:

$$\tfrac{n+2}{n-2} u^{\frac{4}{n-2}} \partial_t u = \triangle_0 u - cR_0 u \qquad \text{on } M \times [0,T],$$
$$u|_{t=0} = u_0 > 0,$$

where $c = c(n) = \frac{n-2}{4(n-1)}$ and $\inf_M u_0 > 0$. We will now use a compactness argument to establish that $u > 0$ for positive time.

By continuity of u and the fact that $\inf_M u_0 > 0$ we have that for each $x \in M$ $\exists \varepsilon(x), t(x)$ such that $u > 0$ on $B_{\varepsilon(x)}(x) \times [0, t(x)]$. The collection $\{B_{\varepsilon(x)}(x)\}_{x \in M}$ is an open cover for M. Thus by compactness of M there exists a finite sub-cover, say $\{B_{\varepsilon(x_i)}(x_i)\}_{i=1}^m$. Letting $\tau = \min_{1 \le i \le m} t(x_i)$ we get $u > 0$ on $M \times [0, \tau]$.

2.5 Uniqueness of Solutions to (2.3)

We know that for $u_0 > 0$ and for T small, the solution u of

$$\frac{n+2}{n-2} u^{\frac{4}{n-2}} \partial_t u = \frac{n+2}{4c(n)} \{\triangle_0 u - cR_0 u\} \qquad \text{on} \quad M \times [0,T], \tag{2.8}$$

$$u|_{t=0} = u_0 > 0,$$

remains positive. Suppose that there exists two positive solutions u, v of this initial value problem (2.8), and let $w := u - v$. Then we have on $M \times [0,T]$

$$\begin{aligned}
\frac{n+2}{4c(n)}\{\triangle_0 w - cR_0 w\} &= \frac{n+2}{n-2} u^{\frac{4}{n-2}} \partial_t u - \frac{n+2}{n-2} v^{\frac{4}{n-2}} \partial_t v \\
&= \frac{n+2}{n-2} u^{\frac{4}{n-2}} \partial_t w + \frac{n+2}{n-2} u^{\frac{4}{n-2}} \partial_t v - \frac{n+2}{n-2} v^{\frac{4}{n-2}} \partial_t v \\
&= \frac{n+2}{n-2} u^{\frac{4}{n-2}} \partial_t w + \frac{n+2}{n-2} \partial_t v (u^{\frac{4}{n-2}} - v^{\frac{4}{n-2}}) \\
&= \frac{n+2}{n-2} u^{\frac{4}{n-2}} \partial_t w + \frac{n+2}{n-2} \partial_t v \int_0^1 \frac{d}{dz} (zu + (1-z)v)^{\frac{4}{n-2}} dz \\
&= \frac{n+2}{n-2} u^{\frac{4}{n-2}} \partial_t w + \frac{4(n+2)}{(n-2)^2} \partial_t v \int_0^1 (zu + (1-z)v)^{\frac{6-n}{n-2}} dz \cdot w.
\end{aligned}$$

Therefore w satisfies

$$\begin{aligned}
\partial_t w &= \frac{n+1}{u^{\frac{4}{n-2}}} \triangle_0 w - \{\frac{4}{(n-2) u^{\frac{4}{n-2}}} \partial_t v \int_0^1 (zu + (1-z)v)^{\frac{6-n}{n-2}} dz + \frac{(n-2)(n+1)R_0}{4(n-1) u^{\frac{4}{n-2}}}\} \cdot w \\
&= a \triangle_0 w - bw \qquad \text{on} \quad M \times [0,T]
\end{aligned}$$

$$w|_{t=0} = 0$$

We note that $a > 0, b$ are continuous functions.

Consider $\tilde{w} := e^{-\lambda t}w$. Then $\partial_t \tilde{w} = -\lambda e^{-\lambda t}w + e^{-\lambda t}\partial_t w = e^{-\lambda t}\{a\triangle_0 w - bw\} - \lambda e^{-\lambda t}w = a\triangle_0 \tilde{w} - (b+\lambda)\tilde{w}$. Thus by choosing $\lambda > 0$ sufficiently large, we have $b + \lambda > 0$. If w is not identically zero on $M \times [0,T]$, then by otherwise interchanging the roles of u and v we may assume that $w > 0$ somewhere. Thus $\tilde{w}$ will have a positive maximum somewhere in $M \times [0,T]$, let (y,τ) denote such a point where the maximum occurs. At (y,τ) we have $\partial_t \tilde{w} \geq 0$ and $\triangle_0 \tilde{w} \leq 0$. Therefore, at (y,τ): $0 \leq \partial_t \tilde{w} = a\triangle_0 \tilde{w} - (b+\lambda)\tilde{w} \leq -(b+\lambda)\tilde{w} < 0$. This is a contradiction. Thus $\tilde{w} \equiv 0 \Rightarrow w \equiv 0 \Rightarrow$ uniqueness.

2.6 Uniqueness of Solutions to (1.2)

Above we have established the uniqueness of solutions to (2.2). We will use this to establish uniqueness of solutions to (1.2). Suppose we have two solutions h, k to the IVP (1.2) for $t \in [0,T]$. Let g be a solution to the unnormalized IVP (2.2) for $t \in [0, T^*]$.

Then there are positive increasing functions of t $\tau(t), \xi(t)$ satisfying $\tau(0) = \xi(0) = 0$ and positive functions of t $\psi(t), \varphi(t)$ satisfying $\psi(0) = \varphi(0) = 1$, so that we have

$$h(x,t) = \psi(\tau(t))g(x,\tau(t)),$$

$$k(x,t) = \varphi(\xi(t))g(x,\xi(t)).$$

We find a function $\eta(t)$ so that $\xi(\eta(t)) = \tau(t)$. Now, $0 < \tau'(t) = \frac{d}{dt}\{\xi(\eta(t))\} = \xi'(\eta(t))\eta'(t) \Rightarrow \eta'(t) = \frac{\tau'(t)}{\xi'(\eta(t))} > 0$. Moreover $0 = \tau(0) = \xi(\eta(0)) \Rightarrow \eta(0) = 0$.

Then,

$$k(x,\eta(t)) = \varphi(\tau(t))g(x,\tau(t)) = \frac{\varphi(\tau(t))}{\psi(\tau(t))}h(x,t).$$

$$\Rightarrow h(x,t) = r(t)k(x,\eta(t)).$$

Moreover, we also have $R_h(x,t) = \frac{R_k(x,\eta(t))}{r(t)}$ and $s_h(t) = \frac{s_k(\eta(t))}{r(t)}$.

Now,

$$h_t(x,t) = \{s_h(t) - R_h(x,t)\}h(x,t)$$

$$= \{s_k(\eta(t)) - R_k(x,\eta(t))\}k(x,\eta(t)).$$

Also,

$$\begin{aligned} h_t(x,t) &= \frac{\partial}{\partial t}\{r(t)k(x,\eta(t))\} \\ &= r'(t)k(x,\eta(t)) + r(t)k_t(x,\eta(t))\eta'(t) \\ &= r'(t)k(x,\eta(t)) + r(t)\{s_k(\eta(t)) - R_k(x,\eta(t))\}k(x,\eta(t))\eta'(t). \end{aligned}$$

Comparing the two, we must have

$$R_k(x,\eta(t))[1 - r(t)\eta'(t)] = s_k(\eta(t))[1 - r(t)\eta'(t)] - r'(t).$$

The left hand side depends on x, and unless $r(t)\eta'(t) = 1$ we must have $R_k(x, \cdot) = s_k(\cdot)$. In this case we will have $r'(t) = 0$ and thus $r(t) \equiv r(0) = 1$. But in this case we also get $h_t, k_t \equiv 0$ and thus $h = k$.

Thus we consider the case where $1 = r(t)\eta'(t)$. But this then forces $r' = 0 \Rightarrow r(t) = Constant$. However, from our definition of r, we see that $r(0) = 1$. Hence we get $\eta'(t) = 1$. But $\eta(0) = 0$, so $\eta(t) = t$. Thus we have established uniqueness of solutions.

Chapter 3

Evolution of the Scalar Curvature

Let $g = u^{\frac{4}{n-2}} g_0$ and $h = v^{\frac{4}{n-2}} g = (uv)^{\frac{4}{n-2}} g_0 = w^{\frac{4}{n-2}} g_0$. Then, $R_h = v^{-\frac{n+2}{n-2}}(\frac{-1}{c}\triangle_g v + R_g v) = w^{-\frac{n+2}{n-2}}(\frac{-1}{c}\triangle_0 w + R_0 w)$. This relationship then gives us the following invariance property:

$$(\triangle_g - cR_g)v = u^{-\frac{n+2}{n-2}}(\triangle_0 - cR_0)(uv).$$

When stated in terms of the Conformal Laplacian $L_g = \triangle_g - cR_g$, we have

$$L_g(v) = u^{-\frac{n+2}{n-2}} L_0(uv)$$

Now assuming $u = u(x,t)$ and $g|_{t=0} = g_0$ we get

$$g_t = \partial_t(u^{\frac{4}{n-2}})g_0 = \frac{4}{n-2}\frac{u_t}{u}u^{\frac{4}{n-2}}g_0 = \frac{4}{n-2}\frac{u_t}{u}g.$$

Taking the time derivative of $R = R_g = u^{-\frac{n+2}{n-2}}(\frac{-1}{c}\triangle_0 u + R_0 u)$ we get

$$\begin{aligned}
R_t &= \frac{\partial}{\partial t}\{u^{-\frac{n+2}{n-2}}(\frac{-1}{c}\triangle_0 u + R_0 u)\} \\
&= -\frac{n+2}{n-2}\frac{u_t}{u}u^{-\frac{n+2}{n-2}}(\frac{-1}{c}\triangle_0 u + R_0 u) + u^{-\frac{n+2}{n-2}}(\frac{-1}{c}\triangle_0 u_t + R_0 u_t) \\
&= -\frac{n+2}{n-2}R\frac{u_t}{u} + u^{-\frac{n+2}{n-2}}(\frac{-1}{c}\triangle_0 u_t + R_0 u_t) \\
&= -\frac{n+2}{n-2}R\frac{u_t}{u} + (\frac{-1}{c}\triangle_g\frac{u_t}{u} + R_g\frac{u_t}{u}) \\
&= -\frac{n+2}{4}R(s-R) + \frac{n-2}{4}(\frac{-1}{c}\triangle_g(s-R) + R(s-R)) \\
&= (n-1)\triangle_g R + R(R-s).
\end{aligned}$$

Next we then compute the evolution equation for the average of the scalar curvature $s = s_g = \frac{1}{V(t)}\int_M R_g dV_g$.

$$\begin{aligned}
s_t &= \frac{\partial}{\partial t}\{\frac{\int R dV}{V}\} \\
&= \frac{1}{V}\int R_t dV + \frac{2n}{(n-2)V}\int R\frac{u_t}{u}dV
\end{aligned}$$

$$= \frac{1}{V}\int (n-1)\triangle R + R(R-s)dV + \frac{n}{2V}\int R(s-R)dV$$

$$= \frac{1}{V}\int R(R-s)dV - \frac{n}{2V}\int R(R-s)dV$$

$$= -\frac{n-2}{2V}\int_M R(R-s)dV_g$$

$$= -\frac{n-2}{2V}(\int_M R(R-s)dV_g - s\int_M R - sdV_g)$$

$$= -\frac{n-2}{2V}\int_M |R-s|^2 dV_g.$$

Thus the evolution equation for $s = s(t)$ is given by $s_t = -\frac{n-2}{2V}\int_M |R-s|^2 dV_g$.

THEOREM 3.0.1 *s is bounded from below.*

Proof: Let $u > 0$ be given. Thus, $Q(M,[g_0]) \leq S(g) = \frac{\int_M R_g dV_g}{V(g)^{\frac{n-2}{n}}}$. Also, since the conformal invariant $Q(M,[g_0]) > -\infty$, we get $-\infty < Q(M,[g_0]) \leq S(g) = (V(g))^{\frac{2}{n}} s_g$. But in the case of the Yamabe flow, $V(g) \equiv V(g^0)$, which we shall call V. Therefore, $s_g \geq \frac{Q(M,[g_0])}{V^{\frac{2}{n}}} > -\infty$.
Q.E.D.

Next we show that $R(\cdot,t)$ is bounded in L^p for all p, uniformly in t. We do this via a sequence of theorems. [1]

THEOREM 3.0.2 *$R(\cdot,t)$ is bounded in $L^{\frac{n}{2}}(M, g(\cdot,t))$ uniformly in t.*

Proof:

$$\frac{d}{dt}\int |R|^{\frac{n}{2}} dV = \frac{d}{dt}\int |R|^{\frac{n}{2}} u^{\frac{2n}{n-2}} dV_0$$

$$= \int \frac{\partial}{\partial t}(|R|^{\frac{n}{2}} u^{\frac{2n}{n-2}})dV_0$$

$$= \int \frac{\partial}{\partial t}(|R|^{\frac{n}{2}})dV + \frac{2n}{n-2}\int |R|^{\frac{n}{2}}\frac{u_t}{u}dV$$

$$= \frac{n}{2}\int \mathrm{sign}(R)|R|^{\frac{n-2}{2}} R_t dV + \frac{n}{2}\int |R|^{\frac{n}{2}}(s-R)dV$$

$$= \frac{n}{2}\int \mathrm{sign}(R)|R|^{\frac{n-2}{2}}((n-1)\triangle R + R(R-s))dV + \frac{n}{2}\int |R|^{\frac{n}{2}}(s-R)dV$$

$$= \frac{n(n-1)}{2}\int \mathrm{sign}(R)|R|^{\frac{n-2}{2}}\triangle R dV$$

[1] Here we will assume that there exists a $C > 0$ such that $\frac{1}{C} \leq u \leq C$. This assumption is justified, since we will demonstrate this later using methods that don't require this result. The following results are needed to show rates of convergence of our solution, after we establish that u is uniformly bounded.

$$= -\frac{n(n-1)}{2}\int < \nabla(\text{sign}(R)|R|^{\frac{n-2}{2}}), \nabla R > dV$$

$$= -\frac{n(n-1)(n-2)}{4}\int |R|^{\frac{n-4}{2}}|\nabla R|^2 dV$$

$$= -\frac{4(n-1)(n-2)}{n}\int |\nabla |R|^{\frac{n}{4}}|^2 dV \leq 0.$$

Q.E.D.

THEOREM 3.0.3 *$R(\cdot,t)$ is bounded in $L^{\frac{n^2}{2n-4}}(M, g(\cdot,t))$ uniformly in t.*

Proof: By theorem 3.0.2, we have

$$\frac{d}{dt}\int |R|^{\frac{n}{2}} dV = -\frac{4(n-1)(n-2)}{n}\int |\nabla |R|^{\frac{n}{4}}|^2 dV.$$

Therefore, for any T we have

$$\int_T^{T+1} \|\nabla |R|^{\frac{n}{4}}\|_{L^2}^2 dt = \int_T^{T+1}\int |\nabla |R|^{\frac{n}{4}}|^2 dV dt$$

$$= \frac{n}{4(n-1)(n-2)}\{\int |R(\cdot,T)|^{\frac{n}{2}} dV_{g(\cdot,T)} - \int |R(\cdot,T+1)|^{\frac{n}{2}} dV_{g(\cdot,T+1)}\}$$

$$\Rightarrow \int_0^T \|\nabla |R|^{\frac{n}{4}}\|_{L^2}^2 dt = \int_0^T \int |\nabla |R|^{\frac{n}{4}}|^2 dV dt$$

$$= \frac{n}{4(n-1)(n-2)}\sum_{\tau=0}^{T-1}\{\int |R(\cdot,\tau)|^{\frac{n}{2}} dV_{g(\cdot,\tau)} - \int |R(\cdot,\tau+1)|^{\frac{n}{2}} dV_{g(\cdot,\tau+1)}\}$$

$$= \frac{n}{4(n-1)(n-2)}\{\int |R(\cdot,0)|^{\frac{n}{2}} dV_{g(\cdot,0)} - \int |R(\cdot,T)|^{\frac{n}{2}} dV_{g(\cdot,T)}\}$$

$$\leq \frac{n}{4(n-1)(n-2)}\int |R(\cdot,0)|^{\frac{n}{2}} dV =: C_1$$

Now, this implies (by way of the Gagliardo-Nirenberg-Sobolev inequality) we have

$$\int_T^{T+1} \||R|^{\frac{n}{4}}\|_{L^2}^2 dt \leq \text{Constant}$$

$$\int_T^{T+1} \|\nabla |R|^{\frac{n}{4}}\|_{L^2}^2 dt \leq \text{Constant}.$$

Hence, the Sobolev inequality yields[2]

[2] One can establish that $\exists C > 0$ such that $\frac{1}{C} \leq u \leq C$ (we will show all of this later). As a result, the metrics $g(\cdot,t)$ are uniformly equivalent for all t. This gives us control of the diameter, the injectivity radius, and the constant in the Sobolev inequality. Note also that the assumption that u is bounded is okay, since the results which we are trying to establish about the boundedness of R are not needed until after the boundedness of u is established.

$$\int_T^{T+1} \|R\|^{\frac{n}{2}}_{L^{\frac{n^2}{2n-4}}} dt = \int_T^{T+1} \||R|^{\frac{n}{4}}\|^2_{L^{\frac{2n}{n-2}}} dt \leq C_2.$$

Now we compute $\frac{d}{dt}\int |R|^p dV$

$$\frac{d}{dt}\|R\|^p_{L^p} = \frac{d}{dt}\int |R|^p dV$$

$$= \frac{d}{dt}\int |R|^p u^{\frac{2n}{n-2}} dV_0$$

$$= \int \partial_t(|R|^p u^{\frac{2n}{n-2}}) dV_0$$

$$= \int \partial_t(|R|^p) dV + \frac{2n}{n-2}\int |R|^p \frac{u_t}{u} dV$$

$$= p\int \operatorname{sign}(R)|R|^{p-1} R_t dV + \frac{n}{2}\int |R|^p(s-R) dV$$

$$= p\int \operatorname{sign}(R)|R|^{p-1}((n-1)\triangle R + R(R-s)) dV + \frac{n}{2}\int |R|^p(s-R) dV$$

$$= p(n-1)\int \operatorname{sign}(R)|R|^{p-1}\triangle R dV + (p-\frac{n}{2})\int |R|^p(R-s) dV$$

$$= -p(n-1)\int < \nabla(\operatorname{sign}(R)|R|^{p-1}), \nabla R > dV + (p-\frac{n}{2})\int |R|^p(R-s) dV$$

$$= -p(p-1)(n-1)\int |R|^{p-2}|\nabla R|^2 dV + (p-\frac{n}{2})\int |R|^p(R-s) dV$$

$$= -\frac{4(n-1)(p-1)}{p}\int |\nabla |R|^{\frac{p}{2}}|^2 dV + (p-\frac{n}{2})\int |R|^p(R-s) dV$$

Now, for $p \geq 2$ we have

$$\||R|^{\frac{p}{2}}\|^2_{W^{1,2}} \leq C\int |\nabla |R|^{\frac{p}{2}}|^2 dV + \||R|^{\frac{p}{2}}\|^2_{L^2},$$

for some $C > 0$ independent of t.

$$\Rightarrow -\int |\nabla |R|^{\frac{p}{2}}|^2 dV \leq \frac{-1}{C}\||R|^{\frac{p}{2}}\|^2_{W^{1,2}} + \frac{1}{C}\||R|^{\frac{p}{2}}\|^2_{L^2}$$

$$\Rightarrow \frac{d}{dt}\|R\|^p_{L^p} \leq -\frac{1}{C}\||R|^{\frac{p}{2}}\|^2_{W^{1,2}} + C\||R|^{\frac{p}{2}}\|^2_{L^2} + C\int |R|^p|R-s| dV$$

$$\leq -\frac{1}{C}\||R|^{\frac{p}{2}}\|^2_{W^{1,2}} + C\||R|^{\frac{p}{2}}\|^2_{L^2} + C\int |R|^{p+1} dV + C\sup|s|\int |R|^p dV$$

$$\leq -\frac{1}{C}\||R|^{\frac{p}{2}}\|^2_{W^{1,2}(M,g(\cdot,t))} + C\|R\|^p_{L^p(M,g(\cdot,t))} + C\|R\|^{p+1}_{L^{p+1}(M,g(\cdot,t))} = A$$

Now, Hölder's inequality implies

$$\int |R|^p dV \le C\Big(\int |R|^{p+1} dV\Big)^{\frac{p}{p+1}}$$

$$\le C \int |R|^{p+1} dV + Constant.$$

Now , using this we get (after possibly increasing C)

$$A \le -\frac{1}{C}\||R|^{\frac{p}{2}}\|^2_{W^{1,2}} + C\|R\|^{p+1}_{L^{p+1}} + C.$$

Now, by Sobolev we have

$$\||R|^{\frac{p}{2}}\|^2_{W^{1,2}} \ge C\||R|^{\frac{p}{2}}\|^2_{L^{\frac{2n}{n-2}}} = C\|R\|^p_{L^{\frac{np}{n-2}}}.$$

Therefore,

$$\frac{d}{dt}\|R\|^p_{L^p} \le -\frac{1}{C}\|R\|^p_{L^{\frac{pn}{n-2}}} + C\|R\|^{p+1}_{L^{p+1}} + C.$$

Next we use the following interpolation inequality:
Assume

$$1 \le s \le r \le t \le \infty, \quad \frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t},$$

and $u \in L^s \cap L^t$, which implies $u \in L^r$ and

$$\|u\|_{L^r} \le \|u\|^{\theta}_{L^s}\|u\|^{1-\theta}_{L^t}.$$

Therefore using $s = q$, $t = \frac{np}{n-2}$ and $r = p+1$ we get

$$\frac{d}{dt}\|R\|^p_{L^p} \le \frac{-1}{C}\|R\|^p_{L^{\frac{pn}{n-2}}} + C\|R\|^{\theta(p+1)}_{L^q}\|R\|^{(1-\theta)(p+1)}_{L^{\frac{pn}{n-2}}} + C.$$

Young's inequality gives us : $ab \le \frac{a^{p'}}{p'} + \frac{b^{q'}}{q'}$, $a, b \ge 0$ and $\frac{1}{p'} + \frac{1}{q'} = 1$. Writing $ab = (p'\varepsilon)^{\frac{1}{p'}} \cdot \frac{b}{(p'\varepsilon)^{\frac{1}{p'}}}$, we then get that $ab \le \varepsilon a^{p'} + \frac{1}{q'(p'\varepsilon)^{\frac{q'}{p'}}} b^{q'}$.
Now, we use this with $a = \|R\|^{(1-\theta)(p+1)}_{L^{\frac{pn}{n-2}}}$ and $b = \|R\|^{\theta(p+1)}_{L^q}$. Also, we require that $p'(1-\theta)(p+1) = p \Rightarrow p' = \frac{p}{(1-\theta)(p+1)}$. Moreover, we need $p' > 1$, so we will require $(1-\theta)(p+1) < p$. Then we get that $q' = \frac{p'}{p'-1} = \frac{p}{\theta(p+1)-1}$.

Thus we get

$$C\|R\|^{\theta(p+1)}_{L^q}\|R\|^{(1-\theta)(p+1)}_{L^{\frac{pn}{n-2}}} \le C\{\varepsilon\|R\|^p_{L^{\frac{pn}{n-2}}} + \frac{1}{q'(p'\varepsilon)^{\frac{q'}{p'}}}\|R\|^{\frac{p\theta(p+1)}{\theta(p+1)-1}}_{L^q}\}.$$

Hence, picking $\varepsilon = \frac{1}{C^2}$

$$\frac{d}{dt}\|R\|^p_{L^p} \le \frac{C^{1+\frac{2q'}{p'}}}{q'(p')^{\frac{q'}{p'}}}\|R\|^{\frac{\theta(p+1)p}{\theta(p+1)-1}}_{L^q} + C,$$

or simply,

$$\frac{d}{dt}\|R\|_{L^p}^p \leq C\|R\|_{L^q}^{\frac{\theta(p+1)p}{\theta(p+1)-1}} + C$$

Note that here we require $q \geq 2$ and θ given by $\theta(\frac{p}{q} - \frac{n-2}{n}) = \frac{p}{p+1} - \frac{n-2}{n}$. Moreover, $\theta(p+1) > 1$ $\Leftrightarrow q > \frac{n}{2}$. Now taking $q = p = \frac{n^2}{2n-4}$ we obtain

$$\frac{d}{dt}\|R\|_{L^{\frac{n^2}{2n-4}}}^{\frac{n^2}{2n-4}} \leq C\|R\|_{L^{\frac{n^2}{2n-4}}}^{\frac{n(n-1)}{n-2}} + C.$$

Letting $y = \|R\|_{L^{\frac{n^2}{2n-4}}}^{\frac{n^2}{2n-4}}$, this is just

$$\frac{d}{dt}y \leq Cy^{\frac{2n-2}{n}} + C.$$

Thus it follows that

$$\begin{aligned}\frac{d}{dt}\log(y+1) &= \frac{\frac{d}{dt}y}{y+1} \\ &\leq \frac{Cy^{\frac{2n-2}{n}} + C}{y+1} \\ &\leq Cy^{\frac{n-2}{n}} + C.\end{aligned}$$

But we have shown that $\forall T$, $\int_T^{T+1} y^{\frac{n-2}{n}} dt \leq C$. In particular, $\exists t \in [T-1, T]$ such that $y(t)^{\frac{n-2}{n}} \leq C$. Therefore,

$$\begin{aligned}\log(y+1)|_T - \log(y+1)|_t &= \int_t^T \frac{d}{d\tau}\log(y+1)d\tau \\ &\leq C\int_t^T y^{\frac{n-2}{n}} + 1 d\tau \\ &\leq \text{Constant}.\end{aligned}$$

Using that $\log(y+1)|_t \leq$ Constant (independent of T), we get our result. Therefore $y \leq C$, for some $C > 0$ independent of t.
Q.E.D.

THEOREM 3.0.4 *The scalar curvature R is bounded in $L^p(M, g(\cdot, t))$ for all $p \geq 2$, and independent of t.*

Proof: From theorem 3.0.3, we know that

$$\frac{d}{dt}\|R\|_{L^p}^p \leq \frac{-1}{C}\|R\|_{L^{\frac{pn}{n-2}}}^p + C\|R\|_{L^q}^{\theta(p+1)}\|R\|_{L^{\frac{pn}{n-2}}}^{(1-\theta)(p+1)} + C$$

for all $p, q \geq 2$. For $q = \frac{n^2}{2n-4}$ we have $\|R\|_{L^q} \leq C$, by the previous theorem.

From this it follows that (here assuming $p + 1 \geq q$ also)

$$\frac{d}{dt}\|R\|_{L^p}^p \leq \frac{-1}{C}\|R\|_{L^{\frac{pn}{n-2}}}^p + C\|R\|_{L^{\frac{pn}{n-2}}}^{(1-\theta)(p+1)} + C.$$

Since $q > \frac{n}{2}$, we have $\theta(p+1) > 1$, and hence $(1-\theta)(p+1) < p$. Young's inequality then gives us

$$\frac{d}{dt}\|R\|_{L^p}^p \leq -\frac{1}{C}\|R\|_{L^{\frac{pn}{n-2}}}^p + C$$

for some constant C.

Therefore $\|R\|_{L^p}$ is bounded.
Q.E.D.

THEOREM 3.0.5 $\|R(\cdot,t)\|_{L^\infty(M,g(\cdot,t))}$ *is bounded, independent of* t.

Proof: Since $V(g) \equiv V < \infty$, for each t fixed we have
$\|R(\cdot,t)\|_{L^\infty(M,g(\cdot,t))} = \lim_{p\to\infty} \|R(\cdot,t)\|_{L^p(M,g(\cdot,t))}$. Also, $\|R(\cdot,t)\|_{L^p(M,g(\cdot,t))} \leq C$, independent of t.
Therefore, we get our result.
Q.E.D.

Chapter 4

Dini Derivatives and Lipschitz Continuity

In the following, we present some notions and results from [8]. If $f(t)$ is a Lipschitz function of t which is not quite differentiable, we say that

$$\frac{df}{dt}(t) \leq c \quad \text{if} \quad \limsup_{h \searrow 0} \frac{f(t+h)-f(t)}{h} \leq c,$$

where we take the lim sup over all forward difference quotients.

Likewise, we say that

$$\frac{df}{dt}(t) \geq c \quad \text{if} \quad \liminf_{h \searrow 0} \frac{f(t+h)-f(t)}{h} \geq c,$$

where we take the lim inf over all forward difference quotients.

Now, for f, g Lipschitz, we say that

$$\frac{df}{dt}(t) \leq \frac{dg}{dt}(t) \quad \text{if} \quad \limsup_{h \searrow 0} \frac{f(t+h)-f(t)}{h} \leq \liminf_{h \searrow 0} \frac{g(t+h)-g(t)}{h},$$

with the above limits taken over all forward difference quotients.

The first result that we shall note is the following lemma

Lemma 4.0.1 *If $f(a) \leq 0$ and $\frac{df}{dt}(t) \leq 0$ at the t-values such that $f(t) \geq 0$, for $a \leq t \leq b$, then $f(b) \leq 0$.*

Proof Let $\varepsilon > 0$, we shall show $f(t) \leq \varepsilon(t-a)$ on $[a, b]$. Let $x \geq a$ be the smallest value of $t \in [a, b]$ such that $f(t) = 0$. Thus if $x > a$ we have $f < 0$ on $[a, x)$, and hence $f(t) \leq \varepsilon(t-a)$ on $[a, x]$. Since $\limsup_{h \searrow 0} \frac{f(x+h)-f(x)}{h} \leq 0$, we have that $\exists \delta > 0$ such that for $h \in [0, \delta]$ $\frac{f(x+h)-f(x)}{h} \leq \varepsilon$. Thus, we have $f(x+h) \leq \varepsilon h \leq \varepsilon(x+h-a)$. Let $a \leq t \leq c$ be the largest such interval with $c \leq b$ such that $f(t) \leq \varepsilon(t-a)$. By continuity of f, $f(t) \leq \varepsilon(t-a)$ on $[a, c]$. If $c < b$, we must have $f(c) > 0$ and thus $\limsup_{h \searrow 0} \frac{f(c+h)-f(c)}{h} \leq 0$. Thus, $\exists \delta > 0$ so that for $h \in [0, \delta]$ we have $\frac{f(c+h)-f(c)}{h} \leq \varepsilon \Rightarrow f(c+h) \leq f(c) + \varepsilon h = \varepsilon(c-a) + \varepsilon h = \varepsilon(c+h-a)$. This is a contradiction, unless $c = b$. Thus $f(t) \leq \varepsilon(t-a) \Rightarrow f(b) \leq \varepsilon(b-a)$. But $\varepsilon > 0$ was arbitrary, hence $f(b) \leq 0$.

Q.E.D.

Corollary 4.0.2 *If $f(a) \geq 0$ and $\frac{df}{dt} \geq 0$ on $[a,b]$, then $f(b) \geq 0$.*

Corollary 4.0.3 *If $f(a) \leq 0$ and $\frac{df}{dt} \leq cf$ on $[a,b]$, then $f(b) \leq 0$.*

Proof: Let $g(t) := e^{-ct}f(t)$, then

$$\frac{g(t+h)-g(t)}{h} = \frac{e^{-c(t+h)}f(t+h) - e^{-ct}f(t)}{h}$$

$$= e^{-ct}\frac{e^{-ch}f(t+h) - f(t)}{h}$$

$$= e^{-ct}\{e^{-ch}\frac{f(t+h)-f(t)}{h} + f(t)\frac{e^{-ch}-1}{h}\}.$$

Thus, we get

$$\limsup_{h\searrow 0}\frac{g(t+h)-g(t)}{h} = \limsup_{h\searrow 0} e^{-ct}\{e^{-ch}\frac{f(t+h)-f(t)}{h} + f(t)\frac{e^{-ch}-1}{h}\}$$

$$\leq e^{-ct}\limsup_{h\searrow 0}\frac{f(t+h)-f(t)}{h} + e^{-ct}f(t)\limsup_{h\searrow 0}\frac{e^{-ch}-1}{h}$$

$$\leq e^{-ct}cf(t) - cf(t)e^{-ct} = 0.$$

Thus, $\frac{dg}{dt} \leq 0 \Rightarrow g(b) \leq 0 \Rightarrow f(b) \leq 0$.
Q.E.D.

Corollary 4.0.4 *If $f(a) \leq g(a)$ and $\frac{df}{dt} \leq \frac{dg}{dt}$ on $[a,b]$, then $f(b) \leq g(b)$.*

Proof: Let $k = f - g$, then $k(a) \leq 0$. Thus we only need to show that $\frac{dk}{dt} \leq 0$ on $[a,b]$.

$$\frac{dk}{dt}(t) \leq 0 \qquad \Longleftrightarrow \qquad \limsup_{h\searrow 0}\{\frac{f(t+h)-f(t)}{h} - \frac{g(t+h)-g(t)}{h}\} \leq 0.$$

However,

$$\limsup_{h\searrow 0}\{\frac{f(t+h)-f(t)}{h} - \frac{g(t+h)-g(t)}{h}\} \leq \limsup_{h\searrow 0}\frac{f(t+h)-f(t)}{h} + \limsup_{h\searrow 0}\{-\frac{g(t+h)-g(t)}{h}\}$$

$$= \limsup_{h\searrow 0}\frac{f(t+h)-f(t)}{h} - \liminf_{h\searrow 0}\frac{g(t+h)-g(t)}{h} \leq 0$$

by assumption.
Q.E.D.

Let $g = g(x,t)$ be a locally Lipschitz function, where $x \in R^k$ or M^k. Let
$f(t) = \sup\{g(x,t) : x \in Y, Y \text{compact}\}$.

Lemma 4.0.5 *$f(t)$ is locally Lipschitz.*

Proof: By compactness of Y, we may choose $y = y(t)$ so that

$$f(t+h) - f(t) = g(y(t+h), t+h) - g(y(t), t)$$

$$\leq g(y(t+h), t+h) - g(y(t+h), t)$$

$$= |g(y(t+h), t+h) - g(y(t+h), t)|$$

$$\leq C|h|,$$

where C is so that $\frac{|g(x,t+h)-g(x,t)|}{|h|} \leq C$ for all $x \in Y$, and all $t \in [0, T]$. Likewise,

$$f(t+h) - f(t) = g(y(t+h), t+h) - g(y(t), t)$$

$$\geq g(y(t), t+h) - g(y(t), t)$$

$$= -|g(y(t), t+h) - g(y(t), t)|$$

$$\geq -C|h|.$$

Therefore,

$$|f(t+h) - f(t)| \leq \sup_{[0,T]} |g_t||h|,$$

where we restrict ourselves to $t \in [0, T]$. Therefore f is locally Lipschitz.
Q.E.D.

Lastly we have

Lemma 4.0.6 $\frac{df}{dt}(t) \leq \sup\{\frac{\partial g}{\partial t}(y,t) : y \in Y(t)\}$, *where* $Y(t) = \{y : g(y,t) = f(t)\}$.

Proof: Choose a sequence of times $\{t_j\}$ with $t_j \searrow t$ for which

$$\lim_{t_j \searrow t} \frac{f(t_j) - f(t)}{t_j - t} = \limsup_{h \searrow 0} \frac{f(t+h) - f(t)}{h}.$$

Choose $y_j \in Y$ with $f(t_j) = g(y_j, t_j)$. By otherwise passing to a subsequence, we may assume $y_j \to y$ for some $y \in Y$. By continuity, $g(y,t) = f(t)$, so $y \in Y(t)$. Since $g(y_j, t) \leq g(y, t)$, we have $f(t_j) - f(t) \leq g(y_j, t_j) - g(y_j, t)$. By the Mean Value Theorem, $\exists T_j$ strictly between t_j and t with $\frac{g(y_j,t_j)-g(y_j,t)}{t_j - t} = \frac{\partial g}{\partial t}(y_j, T_j)$.

Since $T_j \to t$ we also have

$$\lim_{t_j \searrow t} \frac{f(t_j) - f(t)}{t_j - t} \leq \frac{\partial g}{\partial t}(y, t).$$

Q.E.D.

Corollary by replacing g by $-g$ we see that for $f(t) = \inf\{g(y,t) : y \in Y, Y \text{compact}\}$ we get

Corollary 4.0.7 $\frac{df}{dt}(t) \geq \inf\{\frac{\partial g}{\partial t}(y,t) : y \in Y(t)\}$, *where* $Y(t) = \{y : g(y,t) = f(t)\}$.

Chapter 5

Convergence of the Flow in the Scalar Negative Case

We now consider the case where $[g_0]$ is scalar negative. Assume g_0 is a metric so that $R_{g_0} < 0$. Let $g^0 \in [g_0]$ and suppose g satisfies

$$\frac{\partial}{\partial t} g = (s - R)g, \quad \text{on} \quad M \times [0, T^*),$$

$$g|_{t=0} = g^0. \tag{5.1}$$

Writing $g = u^{\frac{4}{n-2}} g_0$, we have that u satisfies

$$\frac{\partial}{\partial t} u^{\frac{4}{n-2}} = \frac{1}{c(n)u} \triangle_{g_0} u - R_{g_0} + s(t) u^{\frac{4}{n-2}}, \tag{5.2}$$

where $c = c(n) = \frac{n-2}{4(n-1)}$.

Define $u_{\min}(t) = \min_{x \in M} u(x,t)$, and $u_{\max}(t) = \max_{x \in M} u(x,t)$. Then the following holds

Lemma 5.0.1

$$\frac{du_{\min}^{\frac{4}{n-2}}}{dt}(t) \geq \min |R_0| + s(t) u_{\min}^{\frac{4}{n-2}}(t), \tag{5.3}$$

and

$$\frac{du_{\max}^{\frac{4}{n-2}}}{dt}(t) \leq \max |R_0| + s(t) u_{\max}^{\frac{4}{n-2}}(t)^1 \tag{5.4}$$

Proof: Let $X(t) = \{x : u_{\max}(t) = u(x,t)\}$ and $x(t) = \{x : u_{\min}(t) = u(x,t)\}$. We get

$$\frac{du_{\min}^{\frac{4}{n-2}}}{dt}(t) \geq \inf\{\frac{\partial}{\partial t} u^{\frac{4}{n-2}}(x,t) : x \in x(t)\}$$

$$= \inf\{\frac{1}{c(n) u_{\min}(t)} \triangle_0 u(x,t) - R_0(x) + s(t) u_{\min}^{\frac{4}{n-2}}(t) : x \in x(t)\}$$

$$\geq \min|R_0| + s(t)u_{\min}^{\frac{4}{n-2}}(t)$$

and

$$\frac{du_{\max}^{\frac{4}{n-2}}}{dt}(t) \leq \sup\{\frac{\partial}{\partial t}u^{\frac{4}{n-2}}(x,t) : x \in X(t)\}$$

$$= \{\frac{1}{c(n)u_{\max}(t)}\triangle_0 u(x,t) - R_0(x) + s(t)u_{\max}^{\frac{4}{n-2}}(t) : x \in X(t)\}$$

$$\leq \max|R_0| + s(t)u_{\max}^{\frac{4}{n-2}}(t).$$

Q.E.D.

Since $s(t) \geq QV(g^0)^{-\frac{2}{n}}$, we can replace lemma 5.0.1 result with

$$\frac{du_{\min}^{\frac{4}{n-2}}}{dt}(t) \geq \min|R_0| - |Q|V^{-\frac{2}{n}}u_{\min}^{\frac{4}{n-2}}(t)$$

where here $V = V(g^0)$ and $g|_{t=0} = g^0$; thus giving us the following lemma

Lemma 5.0.2

$$u_{\min}^{\frac{4}{n-2}}(t) \geq \min\{\frac{\min|R_0|}{2|Q|}V^{\frac{2}{n}}, \frac{u_{\min}^{\frac{4}{n-2}}(0)}{2}\}.$$

Proof: We have for some $C_1, C_2 > 0$ that

$$\frac{d}{dt}u_{\min}^{\frac{4}{n-2}}(t) \geq C_1 - C_2 u_{\min}^{\frac{4}{n-2}}(t)$$

$$\Rightarrow \quad \frac{d}{dt}(e^{C_2 t}u_{\min}^{\frac{4}{n-2}}(t)) \geq C_1 e^{C_2 t} = \frac{d}{dt}(\frac{C_1}{C_2}e^{C_2 t} - \frac{C_1}{C_2} + u_{\min}^{\frac{4}{n-2}}(0))$$

$$\Rightarrow \quad e^{C_2 t}u_{\min}^{\frac{4}{n-2}}(t) \geq \frac{C_1}{C_2}e^{C_2 t} - \frac{C_1}{C_2} + u_{\min}^{\frac{4}{n-2}}(0)$$

$$\Rightarrow \quad u_{\min}^{\frac{4}{n-2}}(t) \geq e^{-C_2 t}u_{\min}^{\frac{4}{n-2}}(0) + \frac{C_1}{C_2}(1 - e^{-C_2 t}).$$

But for $t \leq \frac{\log 2}{C_2}$ we have $e^{-C_2 t} \geq \frac{1}{2} \Rightarrow u_{\min}^{\frac{4}{n-2}}(t) \geq \frac{1}{2}u_{\min}^{\frac{4}{n-2}}(0)$, and for $t > \frac{\log 2}{C_2}$ we have $u_{\min}^{\frac{4}{n-2}}(t) \geq \frac{C_1}{2C_2}$.
Q.E.D.

Then using that s is non-increasing, we get the lemma 5.0.2 can be strengthened to read

$$\frac{du_{\max}^{\frac{4}{n-2}}}{dt}(t) \leq |\min R_0| + s(0)u_{\max}^{\frac{4}{n-2}}(t) \leq |\min R_0| + |s(0)|u_{\max}^{\frac{4}{n-2}}(t).$$

This then gives us the following lemma

Lemma 5.0.3

$$u_{\max}^{\frac{4}{n-2}}(t) \leq u_{\max}^{\frac{4}{n-2}}(0)e^{Bt},$$

$B > 0$ independent of the interval to which we have u certain to exist.

Proof: There are constants $C_1, C_2 > 0$ such that

$$
\begin{aligned}
&\frac{d}{dt}u_{\max}^{\frac{4}{n-2}}(t) \leq C_1 + C_2 u_{\max}^{\frac{4}{n-2}}(t) \\
\Rightarrow \quad &\frac{d}{dt}(e^{-C_2 t}u_{\max}^{\frac{4}{n-2}}(t)) \leq C_1 e^{-C_2 t} = \frac{d}{dt}(-\frac{C_1}{C_2}e^{-C_2 t} + \frac{C_1}{C_2} + u_{\max}^{\frac{4}{n-2}}(0)) \\
\Rightarrow \quad &e^{-C_2 t}u_{\max}^{\frac{4}{n-2}}(t) \leq -\frac{C_1}{C_2}e^{-C_2 t} + \frac{C_1}{C_2} + u_{\max}^{\frac{4}{n-2}}(0) \\
\Rightarrow \quad &u_{\max}^{\frac{4}{n-2}}(t) \leq e^{C_2 t}u_{\max}^{\frac{4}{n-2}}(0) - \frac{C_1}{C_2}(1 - e^{C_2 t}) \leq (u_{\max}^{\frac{4}{n-2}}(0) + \frac{C_1}{C_2})e^{C_2 t}.
\end{aligned}
$$

Q.E.D.

Thus we get that there exists $C > 0$, independent of the interval to which we have that u exists, so that

$$\frac{1}{C} \leq u \leq C e^{\frac{(n-2)B}{4}t}.$$

This then implies that u exists for all time t.

Conservation of volume implies

$$V(g^0) \equiv \int u^{\frac{2n}{n-2}} dV_0 \geq u_{\min}^{\frac{2n}{n-2}}(t) V(g_0)$$

and hence

$$u_{\min}^{\frac{2n}{n-2}} \leq \frac{V(g^0)}{V(g_0)}.$$

Thus if $s \geq 0 \ \forall t$, then lemma 5.0.1 reduces to

$$\frac{du_{\min}^{\frac{4}{n-2}}}{dt}(t) \geq \min|R_0| = \frac{d}{dt}(t \min|R_0| + u_{\min}^{\frac{4}{n-2}}(0))$$

which implies

$$u_{\min}^{\frac{4}{n-2}}(t) \geq u_{\min}^{\frac{4}{n-2}}(0) + t\min|R_0|.$$

Thus we must have $s < 0$ eventually. Therefore we may assume $s(0) < 0$, this then gives us the following refinement of lemma 5.0.1

$$\frac{du_{\max}^{\frac{4}{n-2}}}{dt}(t) \leq \max|R_0| - |s(0)|u_{\max}^{\frac{4}{n-2}}(t).$$

This then gives us the following result.

Lemma 5.0.4

$$u_{\max}^{\frac{4}{n-2}}(t) \leq u_{\max}^{\frac{4}{n-2}}(0) + \frac{\max|R_0|}{|s(0)|},$$

and thus there exists a $C > 0$ *such that* $\frac{1}{C} \leq u \leq C$.

Proof: There are $C_1, C_2 > 0$ such that

$$\frac{du_{\max}^{\frac{4}{n-2}}}{dt}(t) \leq C_1 - C_2 u_{\max}^{\frac{4}{n-2}}(t)$$

$$\Rightarrow \quad \frac{d}{dt}(e^{C_2 t} u_{\max}^{\frac{4}{n-2}}(t)) \leq C_1 e^{C_2 t} = \frac{d}{dt}(\frac{C_1}{C_2} e^{C_2 t} - \frac{C_1}{C_2} + u_{\max}^{\frac{4}{n-2}}(0))$$

$$\Rightarrow \quad u_{\max}^{\frac{4}{n-2}}(t) \leq e^{-C_2 t} u_{\max}^{\frac{4}{n-2}}(0) + \frac{C_1}{C_2}(1 - e^{-C_2 t})$$

$$\leq u_{\max}^{\frac{4}{n-2}}(0) + \frac{C_1}{C_2}.$$

Q.E.D.

Then we proceed to show that $u(x,t)$ converges smoothly at an exponential rate to some function $u_\infty(x)$ as $t \to \infty$, and letting $g_\infty = u_\infty^{\frac{4}{n-2}} g_0$, we have $R_{g_\infty} \equiv$ Constant,which must be negative. To see this, we examine $\frac{\partial}{\partial t} R = (n-1)\triangle R + R(R-s)$.

We define $R_{\min}(t) = \sup_{x \in M} R(x,t)$ and $R_{\max}(t) = \inf_{x \in M} R(x,t)$. Since M is compact, the infimum and supremum are achieved, and let $x(t) = \{x : R_{\min}(t) = R(x,t)\}$ and $X(t) = \{x : R_{\max}(t) = R(x,t)\}$. Also, $\frac{d}{dt}$ of $R_{\max}$ or $R_{\min}$ is in the sense discussed earlier for functions which are not necessarily differentiable, but Lipschitz continuous.

Lemma 5.0.5 $-C \leq R|_{t=0} < -\varepsilon < 0$ *implies*

$$\frac{dR_{\min}}{dt}(t) \geq R_{\min}(t)(R_{\min}(t) - s(t)) \geq s(t)(R_{\min}(t) - s(t)),$$

and

$$\frac{dR_{\max}}{dt}(t) \leq R_{\max}(t)(R_{\max}(t) - s(t)) \leq -\varepsilon(R_{\max}(t) - s(t)).$$

Proof: We have,

$$\frac{dR_{\min}}{dt}(t) \geq \inf\{\frac{\partial R}{\partial t}(x,t) : x \in x(t)\}$$

$$= \inf\{(n-1)\triangle R(x,t) + R_{\min}(t)(R_{\min}(t) - s(t)) : x \in x(t)\}$$

$$\geq R_{\min}(t)(R_{\min}(t) - s(t)),$$

and

$$\frac{dR_{\max}}{dt}(t) \leq \sup\{\frac{\partial R}{\partial t}(x,t) : x \in X(t)\}$$

$$= \sup\{(n-1)\triangle R(x,t) + R_{\max}(t)(R_{\max}(t) - s(t)) : x \in X(t)\}$$

$$\leq R_{\max}(t)(R_{\max}(t) - s(t)).$$

Moreover, $(R_{\min} - s)^2 \geq 0 \Rightarrow R_{\min}(R_{\min} - s) \geq s(R_{\min} - s)$, and thus the first result follows. For the second result, we note that $R_{\max}(0) \leq -\varepsilon$. Thus $R_{\max}$ decreasing and hence $R_{\max} + \varepsilon \leq 0 \Rightarrow (R_{\max} + \varepsilon)(R_{\max} - s) \leq 0 \Rightarrow R_{\max}(R_{\max} - s) \leq -\varepsilon(R_{\max} - s)$.
Q.E.D.

Lemma 5.0.6

$$-C \leq R|_{t=0} < -\varepsilon < 0$$

implies

$$-C \leq R(\cdot, t) \leq -\varepsilon < 0$$

for any t, and thus

$$|s - R| \leq (C - \varepsilon)e^{-\varepsilon t}.$$

Moreover, $g_t \to 0$ at an exponential rate.

Proof:

$$\begin{aligned}
\frac{d}{dt}R_{\max}(t) - \frac{d}{dt}R_{\min}(t) &\leq -\varepsilon(R_{\max}(t) - s(t)) - s(t)(R_{\min}(t) - s(t)) \\
&= -\varepsilon R_{\max}(t) - s(t)R_{\min}(t) + \varepsilon s(t) + s^2(t) \\
&= -\varepsilon R_{\max}(t) + \varepsilon R_{\min}(t) - \varepsilon R_{\min}(t) - s(t)R_{\min}(t) + \varepsilon s(t) + s^2(t) \\
&= -\varepsilon(R_{\max}(t) - R_{\min}(t)) + (s(t) + \varepsilon)(s(t) - R_{\min}(t)) \\
&\leq -\varepsilon(R_{\max}(t) - R_{\min}(t)).
\end{aligned}$$

Hence,

$$\frac{d}{dt}(R_{\max}(t) - R_{\min}(t)) \leq \frac{dR_{\max}}{dt} - \frac{dR_{\min}}{dt}(t) \leq -\varepsilon(R_{\max}(t) - R_{\min}(t)).$$

Thus we are in the situation where we have an $f \geq 0$ and ε 0 so that $\frac{df}{dt}(t) \leq -\varepsilon f(t)$ and we'd like to show that $f(t) \leq f(0)e^{-\varepsilon t}$. Considering $g(t) = e^{\varepsilon t}f(t) - f(0)$, we get $\frac{dg}{dt}(t) \leq 0$ and $g(0) = 0$, thus we have $g(t) \leq 0 \Rightarrow f(t) \leq f(0)e^{-\varepsilon t}$.

Thus

$$R_{\max}(t) - R_{\min}(t) \leq (R_{\max}(0) - R_{\min}(0))e^{-\frac{\varepsilon}{2}t}.$$

Q.E.D.

Recall that there exists $0 < C < \infty$ such that $\frac{1}{C} < u < C^2$. Consider $T > 0$ fixed. We may suppose T is large to insure $s(T) < 0$. Let $x \in M$ and consider $B_R(x)$ so that we also have $s(T - R^2) < 0$.

[2]The Remaining case is to show R eventually becomes negative. We establish this result as follows: $\frac{1}{C} \leq u \leq C$ implies, via results of [11], that we have C^k bounds for $u(\cdot, t)$, independent of t. This then gives C^k bounds for $R - s$ as well. Since $\int (R - s)^2 dV$ must be arbitrarily small for some t-values, and we have H^{2k} bounds on $R - s$, we get (via an interpolation inequality) that $\|R - s\|_{H^k(M \times \{t\})}$ must be arbitrarily small for some large t-values. Then the Sobolev embedding theorem gives that $R - s$ must be arbitrarily small for some large t-values. Since s is negative, this gives that R must become negative.

For

$$L = \sum_{i,j} a^{ij}\partial_{x_i}\partial_{x_j} + \sum_i b^i\partial_{x_i} - c - \partial_t,$$

with

$$\mu|\xi|^2 \leq \sum_{i,j} a^{ij}\xi^i\xi^j \leq \frac{1}{\mu}|\xi|^2$$

for some $0 < \mu < \infty$,

$$\|b\|_\infty \leq \frac{1}{\mu} \quad \text{and} \quad 0 \leq c \leq \frac{1}{\mu}.$$

$$Q(R,(x,t)) := \{(y,s) : y \in B_R(x), t - s < R^2\}.$$

Then, Theorem 4.2 of [11] yields for $u \in W^{1,2}_{n+1}(Q(R,(x,t)))$, and $(x_1,t_1),(x_2,t_2) \in Q(\frac{R}{2},(x,t))$, there exists an $\alpha = \alpha(n,\mu)$ and a $C = C(n,\mu,R)$ such that

$$|u(x_2,t_2) - u(x_1,t_1)| \leq C\{\sup_{Q(R,(x,t))}|u| + \|Lu\|_{L^{n+1}(Q(R,(x,t)))}\}(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\frac{\alpha}{2}}).$$

Where we now apply this theorem to

$$Lu = -u_t + \frac{n-1}{u^{\frac{4}{n-2}}}\triangle_0 u = \frac{(n-1)c(n)R_0}{u^{\frac{6-n}{n-2}}} - (n-1)c(n)s(t)u,$$

for any $x \in M$, $R > 0$ and $t > R^2$.

Thus, there exists a $C = C(R, g_0)$ and an $\alpha = \alpha(n, \sup u, \inf u, g_0) > 0$ so that for $(x_1,t_1),(x_2,t_2) \in Q(\frac{R}{2},(x,t))$ we have

$$|u(x_2,t_2) - u(x_1,t_1)| \leq C(\sup_{Q(R,(x,t))}|u| + \|Lu\|_{L^{n+1}(Q(R,(x,t)))})(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\frac{\alpha}{2}}).$$

Now using that

$$\|Lu\|_{L^\infty}, \|u\|_{L^\infty} < Constant,$$

we get

$$\|u\|_{C^{\frac{\alpha}{2},\alpha}(Q(R,(x,t))} \leq C = C(R)$$

for some $C(R)$. By compactness of M, we then get

$$\|u\|_{C^{\frac{\alpha}{2},\alpha}(M\times[0,T])} \leq C(M, g_0, \sup u, \inf u).$$

Thus,

$$\|u\|_{C^{\frac{\alpha}{2},\alpha}(M\times[0,\infty))} \leq \text{Constant}.$$

Using results of [10] [3] we get control of $\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}}$ assuming we have control of $\|R_0\|_{C^\alpha}$. We define some notation:

$$|u|_{0;U} = [u]_{0;U} = \sup_U |u|,$$

for $k \in \mathbb{N}$

$$[u]_{k;U} = \max_{|\alpha|=k} |D^\alpha u|_{0;U},$$

[3] See Theorem 8.11.1 of [10]

$$|u|_{k;U} = \sum_{j=0}^{k} [u]_{j;U},$$

$$[u]_{\frac{\delta}{2},\delta;U} = \sup_{(x_i,t_i),\in U;(x_1,t_1)\neq(x_2,t_2)} \frac{|u(x_2,t_2)-u(x_1,t_1)|}{|t_2-t_1|^{\frac{\delta}{2}}+|x_2-x_1|^{\delta}},$$

$$|u|_{\frac{\delta}{2},\delta,U} = |u|_{0,U} + [u]_{\frac{\delta}{2},\delta;U}.$$

Let

$$L = \sum_{i,j} a^{ij} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_i b^i \frac{\partial}{\partial x_i} + c - \frac{\partial}{\partial t}.$$

We assume the coefficients of L are all bounded in absolute value by K, and we have the following ellipticity condition

$$\sum_{i,j} a^{ij}\xi_i\xi_j \geq \kappa|\xi|^2.$$

Also, let $\delta \in (0,1)$. Then Theorem 8.11.1 of [10] implies that there exists $N = N(R, K, \delta, \kappa, n)$ such that if $u \in C^{1+\frac{\delta}{2},2+\delta}(Q(3R,(x,t)))$, we have

$$|u|_{1+\frac{\delta}{2},2+\delta;Q(R,(x,t))} \leq N(|Lu|_{\frac{\delta}{2},\delta;Q(2R,(x,t))} + |u|_{0,Q(2R,(x,t))}).$$

Then by applying Theorem 8.12.1 of [10] we get higher regularity results, provided we assume that $\|R_0\|_{C^{k+\alpha}(M,g_0)}$ is under control. Now for completeness we state this theorem.

Theorem 8.12.1 of [10] 5.0.7 *Let L be as above. Assume for $R > 0$, and $k \in \mathbb{N}\cup\{0\}$, we have*

$$|D^\alpha a^{ij}|_{\frac{\delta}{2},\delta;Q(2R,(x,t))}, \quad |D^\alpha b^i|_{\frac{\delta}{2},\delta;Q(2R,(x,t))} \quad \text{and} \quad |D^\alpha c|_{\frac{\delta}{2},\delta;Q(2R,(x,t))} \leq K,$$

for all multi-indices $\alpha \in \mathbb{R}^n$, $|\alpha| \leq k$. Assume $u \in C^{1+\frac{\delta}{2},2+\delta}(Q(2R,(x,t))$ and $D^\alpha(Lu) \in C^{\frac{\delta}{2},\delta}(Q(2R,(x,t))$ for $|\alpha| \leq k$. Then $D^\alpha u \in C^{1+\frac{\delta}{2},2+\delta}(Q(R,(x,t))$ and $\exists N = N(R,k,K,\kappa,\delta,n)$ such that

$$\sum_{|\alpha|\leq k} |D^\alpha u|_{1+\frac{\delta}{2},2+\delta;Q(R,(x,t))} \leq N\{\sum_{|\alpha|\leq k} |D^\alpha(Lu)|_{\frac{\delta}{2},\delta;Q(2R,(x,t))} + |u|_{0;Q(2R,(x,t))}\}.$$

Therefore, $\|u\|_{C^{k,2k}(M\times[T,\infty))} \leq C$, C independent of T, which then gives us control on $\|R-s\|_{H^{2k}(M\times\{T\})} \leq C$, for C independent of T.

Now, since

$$\int_0^\infty \int_M (R-s)^2 dV dt < \infty,$$

we get

$$\int_0^\infty \int_M (R-s)^2 dV_0 dt < \infty$$

since we have $\frac{1}{C} < u < C$. Thus, given $\varepsilon > 0$, we can find T so that $\int_T^\infty \int_M (R-s)^2 dV_0 dt \leq \varepsilon$. Thus, there exists a $t \in [T, T+1]$ so that

$$\int_M (R(\cdot,t) - s(t))^2 dV_0 \leq \varepsilon.$$

Now, we apply Theorem 3.70 of [1] which states:

Theorem 3.70 of [1] 5.0.8 *For M^n a compact manifold without boundary, let $1 \leq q, r \leq \infty$ be real numbers, $0 \leq j < m$ be integers and $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$. Then $\exists K = K(n,m,j,q,r,a,M)$ so that $\forall f$ with $\int f dV = 0$, we have*

$$\|\nabla^j f\|_{L^p} \leq K \|\nabla^m f\|_{L^r}^a \|f\|_{L^q}^{1-a}.$$

We apply this theorem to $f = R - s$, $a = \frac{1}{2}$, $p = q = r = 2$, $j = k$ and $m = 2k$. to get

$$\|R - s\|_{H^k(M \times \{t\}, g_0)} \leq C\sqrt{\varepsilon}.$$

Since C is independent of t and we only need k so that H^k embeds into C^0, we can pick ε, t so that the right hand side is less than $\frac{|s(0)|}{2B}$ with B the constant from the Sobolev embedding theorem for the embedding of $H^k \to C^0$. Thus getting that $R(\cdot,t) < \frac{s(0)}{2}$. Thus R is strictly negative at time t, and thus our convergence results established earlier now apply to get exponential convergence of $|R - s| \to 0$. Thus applying the results of [11], we get exponential convergence of $R - s$ with respect to any C^k norm.

Chapter 6

Convergence of the Flow in the Scalar Flat Case

Thus we are now assuming that g_0 is a metric with $R_{g_0} \equiv 0$. Let $g^0 \in [g_0]$ and suppose g satisfies

$$\frac{\partial}{\partial t} g = (s - R)g, \quad \text{on} \quad M \times [0, T^*),$$

$$g|_{t=0} = g^0. \tag{6.1}$$

Writing $g = u^{\frac{4}{n-2}} g_0$, we may assume u satisfies

$$\frac{\partial}{\partial t}\{u^{\frac{n+2}{n-2}}\} = \frac{n+2}{4c(n)}\{\triangle_{g_0} u + c(n)s(t)u^{\frac{n+2}{n-2}}\}$$

with $u|_{t=0}$ such that $u(\cdot, 0)^{\frac{4}{n-2}} g_0 = g^0$. We have $c = c(n) = \frac{n-2}{4(n-1)}$ and $L_{g_0} u = \triangle_{g_0} u - cR_{g_0} u = \triangle_{g_0} u$, with the last equality holding by the scalar flat assumption.

Lemma 6.0.1

$$s \geq 0$$

for all time.

Proof: $Q = 0 \Rightarrow s \geq 0$.
Q.E.D.

Lemma 6.0.2

$$\frac{du_{\min}^{\frac{n+2}{n-2}}}{dt}(t) \geq \frac{n+2}{4} s(t) u_{\min}^{\frac{n+2}{n-2}}(t),$$

and

$$\frac{du_{\max}^{\frac{n+2}{n-2}}}{dt}(t) \leq \frac{n+2}{4} s(t) u_{\max}^{\frac{n+2}{n-2}}(t).$$

Proof: Using that $x(t)$ is the set of points in $M \times \{t\}$ where $u_{\min}$ is obtained and $X(t)$ is the set of points in $M \times \{t\}$ where $u_{\max}$ is obtained, we have

$$\frac{du_{\min}^{\frac{n+2}{n-2}}}{dt}(t) \geq \inf\{\partial_t(u^{\frac{n+2}{n-2}})(x,t) : x \in x(t)\}$$
$$= \inf\{\frac{n+2}{4c(n)}\{\triangle_0 u(x,t) + c(n)s(t)u_{\min}^{\frac{n+2}{n-2}}(t)\} : x \in x(t)\}$$
$$\geq \frac{n+2}{4}s(t)u_{\min}^{\frac{n+2}{n-2}}(t),$$

and

$$\frac{du_{\max}^{\frac{n+2}{n-2}}}{dt}(t) \leq \sup\{\partial_t(u^{\frac{n+2}{n-2}})(x,t) : x \in X(t)\}$$
$$= \sup\{\frac{n+2}{4c(n)}\{\triangle_0 u(x,t) + c(n)s(t)u_{\max}^{\frac{n+2}{n-2}}(t)\} : x \in X(t)\}$$
$$\leq \frac{n+2}{4}s(t)u_{\max}^{\frac{n+2}{n-2}}(t).$$

Q.E.D.

This then implies the following theorem

Theorem 6.0.3 *A solution u exists $\forall t \geq 0$.*

Proof: By the above lemma,

$$\frac{du_{\min}^{\frac{n+2}{n-2}}}{dt}(t) \geq \frac{n+2}{4}s(t)u_{\min}^{\frac{n+2}{n-2}}(t) \geq 0 = \frac{d}{dt}(u_{\min}^{\frac{n+2}{n-2}}(0)).$$

Thus,

$$u_{\min}^{\frac{n+2}{n-2}}(t) \geq u_{\min}^{\frac{n+2}{n-2}}(0).$$

Also,

$$\frac{du_{\max}^{\frac{n+2}{n-2}}}{dt}(t) \leq \frac{n+2}{4}s(t)u_{\max}^{\frac{n+2}{n-2}}(t) \leq \frac{n+2}{4}s(0)u_{\max}^{\frac{n+2}{n-2}}(t).$$

Now, by considering $g(t) = e^{-\frac{n+2}{4}s(0)t}u_{\max}^{\frac{n+2}{n-2}}(t) - u_{\max}^{\frac{n+2}{n-2}}(0)$, we get $\frac{dg}{dt} \leq 0$ and $g(0) = 0$. Thus, $g(t) \leq 0 \Rightarrow u_{\max}^{\frac{n+2}{n-2}}(t) \leq e^{\frac{n+2}{4}s(0)t}u_{\max}^{\frac{n+2}{n-2}}(0)$. Therefore,

$$u_{\min}(0) \leq u(x,t) \leq u_{\max}(0)e^{\frac{n-2}{4}s(0)t},$$

and thus our solution u exists for all time.
Q.E.D.

Theorem 6.0.4

$$u_{\max}(t) \leq \frac{u_{\max}(0)}{u_{\min}(0)}u_{\min}(t)$$

Proof: Let's define

$$f(t) := (\frac{u_{\max}(t)}{u_{\min}(t)})^{\frac{n+2}{n-2}} - (\frac{u_{\max}(0)}{u_{\min}(0)})^{\frac{n+2}{n-2}}.$$

Now for $h > 0$ we have

$$\frac{f(t+h)-f(t)}{h} = \frac{1}{h}\{\frac{u_{\max}^{\frac{n+2}{n-2}}(t+h)}{u_{\min}^{\frac{n+2}{n-2}}(t+h)} - \frac{u_{\max}^{\frac{n+2}{n-2}}(t)}{u_{\min}^{\frac{n+2}{n-2}}(t)}\}$$

$$= \frac{1}{h}\frac{u_{\min}^{\frac{n+2}{n-2}}(t)u_{\max}^{\frac{n+2}{n-2}}(t+h) - u_{\min}^{\frac{n+2}{n-2}}(t+h)u_{\max}^{\frac{n+2}{n-2}}(t)}{u_{min}^{\frac{n+2}{n-2}}(t+h)u_{\min}^{\frac{n+2}{n-2}}(t)}$$

$$= \frac{1}{h}\{\frac{u_{\min}^{\frac{n+2}{n-2}}(t)u_{\max}^{\frac{n+2}{n-2}}(t+h) - u_{\min}^{\frac{n+2}{n-2}}(t)u_{\max}^{\frac{n+2}{n-2}}(t)}{u_{min}^{\frac{n+2}{n-2}}(t+h)u_{\min}^{\frac{n+2}{n-2}}(t)} - \frac{u_{\max}^{\frac{n+2}{n-2}}(t)u_{\min}^{\frac{n+2}{n-2}}(t+h) - u_{\min}^{\frac{n+2}{n-2}}(t)u_{\max}^{\frac{n+2}{n-2}}(t)}{u_{min}^{\frac{n+2}{n-2}}(t+h)u_{\min}^{\frac{n+2}{n-2}}(t)}\}$$

$$= \frac{1}{u_{\min}^{\frac{n+2}{n-2}}(t+h)}\frac{u_{\max}^{\frac{n+2}{n-2}}(t+h) - u_{\max}^{\frac{n+2}{n-2}}(t)}{h} - \frac{u_{\max}^{\frac{n+2}{n-2}}(t)}{u_{\min}^{\frac{n+2}{n-2}}(t+h)u_{\min}^{\frac{n+2}{n-2}}(t)}\frac{u_{\min}^{\frac{n+2}{n-2}}(t+h) - u_{\min}^{\frac{n+2}{n-2}}(t)}{h}$$

Thus,

$$\limsup_{h\searrow 0}\frac{f(t+h)-f(t)}{h}$$

$$= \limsup_{h\searrow 0}\{\frac{1}{u_{\min}^{\frac{n+2}{n-2}}(t+h)}\frac{u_{\max}^{\frac{n+2}{n-2}}(t+h) - u_{\max}^{\frac{n+2}{n-2}}(t)}{h} - \frac{u_{\max}^{\frac{n+2}{n-2}}(t)}{u_{\min}^{\frac{n+2}{n-2}}(t+h)u_{\min}^{\frac{n+2}{n-2}}(t)}\frac{u_{\min}^{\frac{n+2}{n-2}}(t+h) - u_{\min}^{\frac{n+2}{n-2}}(t)}{h}\}$$

$$\leq \limsup_{h\searrow 0}\frac{1}{u_{\min}^{\frac{n+2}{n-2}}(t+h)}\frac{u_{\max}^{\frac{n+2}{n-2}}(t+h) - u_{\max}^{\frac{n+2}{n-2}}(t)}{h} - \liminf_{h\searrow 0}\frac{u_{\max}^{\frac{n+2}{n-2}}(t)}{u_{\min}^{\frac{n+2}{n-2}}(t+h)u_{\min}^{\frac{n+2}{n-2}}(t)}\frac{u_{\min}^{\frac{n+2}{n-2}}(t+h) - u_{\min}^{\frac{n+2}{n-2}}(t)}{h}$$

$$\leq \frac{1}{u_{\min}^{\frac{n+2}{n-2}}(t)}\frac{du_{\max}^{\frac{n+2}{n-2}}}{dt}(t) - \frac{u_{\max}^{\frac{n+2}{n-2}}(t)}{(u_{\min}^{\frac{n+2}{n-2}}(t))^2}\frac{du_{\min}^{\frac{n+2}{n-2}}}{dt}(t)$$

$$\leq \frac{1}{u_{\min}^{\frac{n+2}{n-2}}(t)}\frac{n+2}{4}s(t)u_{\max}^{\frac{n+2}{n-2}}(t) - \frac{u_{\max}^{\frac{n+2}{n-2}}(t)}{(u_{\min}^{\frac{n+2}{n-2}}(t))^2}\frac{n+2}{4}s(t)u_{\min}^{\frac{n+2}{n-2}}(t)$$

$$= 0.$$

Therefore, $\frac{df}{dt}(t) \leq 0$. Moreover, $f(0) \leq 0 \Rightarrow f(t) \leq 0 \Rightarrow (\frac{u_{\max}(t)}{u_{min}(t)})^{\frac{n+2}{n-2}} \leq (\frac{u_{\max}(0)}{u_{min}(0)})^{\frac{n+2}{n-2}} \Rightarrow$ $u_{\max}(t) \leq \frac{u_{\max}(0)}{u_{\min}(0)}u_{\min}(t)$.
Q.E.D.

Corollary 6.0.5 *There exists $C > 0$ such that*

$$\frac{1}{C} \leq u \leq C.$$

Proof: By theorem 6.0.4, we have

$$u_{\max}(t) \leq \frac{u_{\max}(0)}{u_{\min}(0)} u_{\min}(t).$$

However, conservation of volume implies

$$V(g^0) = \int_M u^{\frac{2n}{n-2}} dV_0 \geq u_{\min}^{\frac{2n}{n-2}} V(g_0).$$

Thus,

$$u_{\min}(t) \leq (\frac{V(g^0)}{V(g_0)})^{\frac{n-2}{2n}}.$$

Moreover,

$$u_{\max}(t) \leq \frac{u_{\max}(0)}{u_{\min}(0)} (\frac{V(g^0)}{V(g_0)})^{\frac{n-2}{2n}}.$$

Q.E.D.

Next we show that we actually have $u(\cdot,t) \to u_\infty(\cdot) \equiv$ Constant, smoothly at an exponential rate, and thus $g \to g_\infty := u_\infty^{\frac{4}{n-2}} g_0$. Moreover, $R_{g_\infty} \equiv 0$ then follows since $u_\infty \equiv$ Constant. We will do this by establishing the following:

1. $\|\nabla_0 u(\cdot,t)\|_{L^2} \leq Ce^{-At}$, for some $C, A > 0 \Rightarrow s(t) \leq Ce^{-At}$ for some $C, A > 0$.

2. $\int_T^\infty \int (\triangle_0 u)^2 dV_0 dt \leq Ce^{-AT}$, for some $C, A > 0$.

3. $\int u^{\frac{n+2}{n-2}} dV_0 \to L$ at an exponential rate (in t) , for some L.

We remark that **1-3** follow directly from the PDE. Then **3** and Poincaré imply **4**.

4. $\|u^{\frac{n+2}{n-2}}(\cdot,t) - \frac{L}{V(g_0)}\|_{L^2(M,g_0)} \leq Ce^{-At}$, for some $C, A > 0$.

5. The results established in the previous section, namely $\frac{1}{C} \leq u \leq C$ giving C^∞ bounds on u by way of the arguments of [10] and [11], imply that we have control of $\|u^{\frac{n+2}{n-2}}\|_{H^{2k}(M\times\{t\})}$, independent of t. Thus, we have control of $\|u^{\frac{n+2}{n-2}} - \frac{L}{V(g_0)}\|_{H^{2k}(M\times\{t\})}$, independent of t. This and **4**, along with the interpolation inequality in [1], give us that $\|u^{\frac{n+2}{n-2}} - \frac{L}{V(g_0)}\|_{H^k(M\times\{t\})} \leq Ce^{-At}$ for some $C, A > 0$.

Now we carry out the explicit details for **1-4** to then get the convergence by way of the argument described in **5**. We note that this is sufficient, since once we establish **5**, we get that $u^{\frac{4}{n-2}}$ converges smoothly to a constant at an exponential rate. Let $u_\infty^{\frac{4}{n-2}}$ denote such a constant. Then we clearly have that $g \to g_\infty := u_\infty^{\frac{4}{n-2}} g_0$ at an exponential rate. Since g_∞ is a constant multiple of g_0 we have $R_{g_\infty} \equiv 0$.

Lemma 6.0.6 *$s \to 0$ as $t \to \infty$.*

Proof: If $s \geq C > 0$ for some constant C, then

$$\frac{du_{\min}^{\frac{n+2}{n-2}}}{dt}(t) \geq C \cdot \frac{n+2}{4} \cdot u_{\min}^{\frac{n+2}{n-2}}(t)$$

which implies

$$\frac{du_{\min}^{\frac{n+2}{n-2}}}{dt}(t) - C \cdot \frac{n+2}{4} \cdot u_{\min}^{\frac{n+2}{n-2}}(t) \geq 0.$$

Letting $g(t) = e^{-\frac{n+2}{4}Ct} u_{\min}^{\frac{n+2}{n-2}}(t) - u_{\min}^{\frac{n+2}{n-2}}(0)$, we get $\frac{dg}{dt}(t) \geq 0$ and $g(0) = 0$. Thus, $g(t) \geq 0 \Rightarrow u_{\min}^{\frac{n+2}{n-2}}(t) \geq e^{\frac{n+2}{4}Ct} u_{\min}^{\frac{n+2}{n-2}}(0)$. But this is a contradiction, since $u_{\min}$ is bounded from above. Thus $s \to 0$ as $t \to \infty$.
Q.E.D.

Next we proceed to show that we actually have exponential convergence.

$$\frac{n+2}{n-2} u^{\frac{4}{n-2}} \frac{\partial u}{\partial t} = \frac{n+2}{4c(n)} \{\triangle_0 u + c(n)s(t) u^{\frac{n+2}{n-2}}\}$$

which implies

$$\frac{1}{n-1} \triangle_0 u \frac{\partial u}{\partial t} = u^{-\frac{4}{n-2}} (\triangle_0 u)^2 + csu\triangle_0 u$$

$$\frac{1}{n-1} \int \triangle_0 u \frac{\partial u}{\partial t} dV_0 = \int u^{-\frac{4}{n-2}} (\triangle_0 u)^2 dV_0 + cs \int u \triangle_0 u dV_0.$$

However,

$$\begin{aligned} \frac{1}{n-1} \int \triangle_0 u \frac{\partial u}{\partial t} dV_0 &= -\frac{1}{n-1} \int \nabla_0 u \cdot \nabla_0 \partial_t u dV_0 \\ &= -\frac{1}{2(n-1)} \int \partial_t |\nabla_0 u|^2 dV_0 \\ &= -\frac{1}{2(n-1)} \frac{d}{dt} \int |\nabla_0 u|^2 dV_0, \end{aligned}$$

and

$$\int u \triangle_0 u dV_0 = -\int |\nabla_0 u|^2 dV_0.$$

Therefore,

$$\frac{1}{n-1} \frac{d}{dt} \int |\nabla_0 u|^2 dV_0 = -2 \int u^{-\frac{4}{n-2}} (\triangle_0 u)^2 dV_0 + 2cs \int |\nabla_0 u|^2 dV_0.$$

Lemma 6.0.7

$$\|\nabla_0 u\|_{L^2(M,g_0)} \leq \hat{C} \|\triangle_0 u\|_{L^2(M,g_0)}$$

for some $\hat{C} > 0$ *independent of* u.

Proof: We know that $\exists C > 0$ such that

$$\|\nabla u\|_{L^2} \leq C(\|\triangle u\|_{L^2} + \|u\|_{L^2}).$$

Suppose this lemma is not true. Then $\exists \{u_j\}$ such that

$$j\|\triangle u_j\|_{L^2} \leq \|\nabla u_j\|_{L^2} \leq C(\|\triangle u_j\|_{L^2} + \|u_j\|_{L^2}).$$

We may also require that $\|\nabla u_j\|_{L^2} = 1\ \forall j$. Hence, we get that $\|\triangle u_j\|_{L^2} \to 0$ as $j \to \infty$.

Now, Poincaré's inequality implies that

$$\|u_j - \bar{u}_j\|_{L^2} \leq C\|\nabla u_j\|_{L^2} \leq C.$$

Hence, $v_j = u_j - \bar{u}_j$ converges weakly in H^2 along a subsequence, which we may assume is our original sequence. Thus, Sobolev embedding implies that $v_j \to v$ in the H^1 sense, for some v. Also, $\nabla v = 0$ which implies v is constant, and hence $v \equiv 0$. But $\|\nabla v_j\|_{L^2} \equiv 1$, and thus we get a contradiction.
Q.E.D.

This then gives us

$$\int u^{-\frac{4}{n-2}}(\triangle_0 u)^2 dV_0 \geq \frac{1}{u_{\max}^{\frac{4}{n-2}}(t)} \int (\triangle_0 u)^2 dV_0$$

$$\geq (\frac{V(g_0)}{V(g^0)})^{\frac{2}{n}} \frac{u_{\min}^{\frac{4}{n-2}}(0)}{u_{\max}^{\frac{4}{n-2}}(0)} \int (\triangle_0 u)^2 dV_0$$

$$\geq (\frac{V(g_0)}{V(g^0)})^{\frac{2}{n}} \frac{u_{\min}^{\frac{4}{n-2}}(0)}{\hat{C}^2 u_{\max}^{\frac{4}{n-2}}(0)} \int |\nabla_0 u|^2 dV_0 = C \int |\nabla_0 u|^2 dV_0.$$

Putting this all together, we then get

$$\frac{1}{n-1}\frac{d}{dt}\int |\nabla_0 u|^2 dV_0 \leq (2c(n)s(t) - 2C)\int |\nabla_0 u|^2 dV_0.$$

Thus,

$$\frac{d}{dt}\log \int |\nabla_0 u|^2 dV_0 = \frac{\frac{d}{dt}\int |\nabla_0 u|^2 dV_0}{\int |\nabla_0 u|^2 dV_0} \leq (n-1)(2cs(t) - 2C) \leq -A$$

if t is sufficiently large, where $A > 0$ is a constant. Thus we get

$$\int |\nabla_0 u(\cdot, t)|^2 dV_0 \leq \int |\nabla_0 u(\cdot, 0)|^2 dV_0 \cdot e^{-At} = Be^{-At}.$$

We may make this assumption since $s \to 0$ as $t \to \infty$. This then actually implies that

$$s(t) = \frac{\int |\nabla_0 u|^2 dV_0}{c(n)V} \leq \frac{B}{c(n)V}e^{-At},$$

for t sufficiently large, where $V = \int_M u^{\frac{2n}{n-2}} dV_0 = V(g^0)$. Once again consider,

$$\frac{1}{n-1}\frac{d}{dt}\int |\nabla_0 u|^2 dV_0 + 2\int (\triangle_0 u)^2 u^{-\frac{4}{n-2}} dV_0 = 2c(n)s\int |\nabla_0 u|^2 dV_0.$$

Integrating with respect to time we then get

$$\frac{1}{n-1}\int_T^\infty \frac{d}{dt}\int |\nabla_0 u|^2 dV_0 dt + 2\int_T^\infty \int (\triangle_0 u)^2 u^{-\frac{4}{n-2}} dV_0 dt = 2c(n)\int_T^\infty s(t)\int |\nabla_0 u|^2 dV_0 dt$$

which implies

$$2\int_T^\infty \int (\triangle_0 u)^2 u^{-\frac{4}{n-2}} dV_0 dt = \frac{1}{n-1}\int |\nabla_0 u(\cdot,T)|^2 dV_0 + 2c\int_T^\infty s(t)\int |\nabla_0 u|^2 dV_0 dt$$

$$\leq \frac{1}{n-1}Be^{-AT} + 2\frac{B}{V}e^{-AT}\int_T^\infty \int |\nabla_0 u|^2 dV_0 dt$$

$$\leq \frac{1}{n-1}Be^{-AT} + 2\frac{B^2}{AV}e^{-2AT}$$

$$\leq He^{-AT},$$

for some constant $H > 0$.
Recalling that $R = -\frac{1}{c}\frac{L_0 u}{u^{\frac{n+2}{n-2}}} = -\frac{1}{c}\frac{\triangle_0 u}{u^{\frac{n+2}{n-2}}}$, we get

$$\int_T^\infty \int R^2 dV dt = \int_T^\infty \int R^2 u^{\frac{2n}{n-2}} dV_0 dt$$

$$= \int_T^\infty \int \frac{1}{c^2}(\triangle_0 u)^2 u^{-\frac{4}{n-2}} dV_0 dt$$

$$\leq \frac{H}{2c^2}e^{-AT}.$$

Therefore, for all T there exists a $t \in [T, T+1]$ such that $\int R^2 dV \leq \frac{H}{2c^2}e^{-At}$. Moreover,

$$s(t) = \frac{\int RdV}{V} \leq \frac{1}{\sqrt{V}}\sqrt{\int R^2 dV} \leq \sqrt{\frac{H}{2c^2 V}}e^{-\frac{A}{2}t}.$$

But s is decreasing, thus for every t-value this inequality holds.

Next we will proceed to show $\int u^{\frac{n+2}{n-2}} dV_0$ converges exponentially. Recall that

$$\frac{\partial}{\partial t}u^{\frac{n+2}{n-2}} = \frac{n+2}{4c(n)}\{\triangle_0 u + c(n)s(t)u^{\frac{n+2}{n-2}}\}.$$

Thus,

$$\frac{d}{dt}\int u^{\frac{n+2}{n-2}} dV_0 = \int \frac{\partial}{\partial t}u^{\frac{n+2}{n-2}} dV_0$$

$$= \frac{n+2}{4c(n)}\int \triangle_0 u dV_0 + c(n)s(t)\int u^{\frac{n+2}{n-2}} dV_0$$

$$= \frac{n+2}{4}s(t)\int u^{\frac{n+2}{n-2}} dV_0$$

$$\leq \frac{n+2}{4c(n)}\sqrt{\frac{H}{2V}}e^{-\frac{A}{2}t}\int u^{\frac{n+2}{n-2}} dV_0.$$

Therefore,

$$\frac{d}{dt}\int u^{\frac{n+2}{n-2}} dV_0 \leq \frac{n+2}{4c(n)}\sqrt{\frac{H}{2V}}e^{-\frac{A}{2}t}\int u^{\frac{n+2}{n-2}} dV_0.$$

Thus,

$$\frac{d}{dt}\log\int u^{\frac{n+2}{n-2}}dV_0 \leq \frac{n+2}{4c(n)}\sqrt{\frac{H}{2V}}e^{-\frac{A}{2}t}$$

which implies

$$\log\frac{\int u^{\frac{n+2}{n-2}}(\cdot,t)dV_0}{\int u^{\frac{n+2}{n-2}}(\cdot,0)dV_0} \leq \frac{n+2}{2Ac(n)}\sqrt{\frac{H}{2V}}(1-e^{-\frac{A}{2}t}) \leq \frac{n+2}{2Ac(n)}\sqrt{\frac{H}{2V}}$$

and thus

$$\int u^{\frac{n+2}{n-2}}(\cdot,t)dV_0 \leq \int u^{\frac{n+2}{n-2}}(\cdot,0)dV_0 \cdot e^{\frac{n+2}{2Ac(n)}\sqrt{\frac{H}{2V}}}.$$

Thus $\int u^{\frac{n+2}{n-2}}dV_0$ is bounded from above which implies $\frac{d}{dt}\int u^{\frac{n+2}{n-2}}dV_0 \leq Ce^{-At}$ for some $C, A > 0$. Moreover, $\int u^{\frac{n+2}{n-2}}dV_0$ is non-decreasing. Thus, given any $t_2 > t_1$, we get

$$\begin{aligned}|\int u^{\frac{n+2}{n-2}}(\cdot,t_2)dV_0 - \int u^{\frac{n+2}{n-2}}(\cdot,t_1)dV_0| &= \int u^{\frac{n+2}{n-2}}(\cdot,t_2)dV_0 - \int u^{\frac{n+2}{n-2}}(\cdot,t_1)dV_0 \\ &\leq \frac{C}{A}(e^{-At_1} - e^{-At_2}).\end{aligned}$$

Thus, $\int u^{\frac{n+2}{n-2}}dV_0 \nearrow L$ as $t \to \infty$, for some constant L.

Poincaré's inequality implies that there exists $C = C(M, g_0)$ such that

$$\|u^{\frac{n+2}{n-2}} - \frac{1}{V(g^0)}\int u^{\frac{n+2}{n-2}}dV_0\|^2_{L^2(M,g_0)} \leq C\|\nabla_0 u\|^2_{L^2(M,g_0)} \leq C\cdot Be^{-At}.$$

Moreover, from the above calculations we get that $\int u^{\frac{n+2}{n-2}}dV_0$ converges at an exponential rate, and let L denote its limit. We also get that

$$|\int u^{\frac{n+2}{n-2}}dV_0 - L| \leq C'e^{-\frac{A}{2}t}$$

for some positive C'. Thus,

$$\begin{aligned}\|u^{\frac{n+2}{n-2}} - \frac{L}{V(g^0)}\|_{L^2} &\leq \|u^{\frac{n+2}{n-2}} - \frac{1}{V(g^0)}\int u^{\frac{n+2}{n-2}}dV_0\|_{L^2} + \frac{1}{V(g^0)}\|\int u^{\frac{n+2}{n-2}}dV_0 - L\|_{L^2} \\ &\leq \sqrt{CB}e^{-\frac{A}{2}t} + \frac{1}{\sqrt{V(g^0)}}|\int u^{\frac{n+2}{n-2}}dV_0 - L| \\ &\leq \sqrt{CB}e^{-\frac{A}{2}t} + \frac{C'}{\sqrt{V(g^0)}}e^{-\frac{A}{2}t}.\end{aligned}$$

We then get that $u^{\frac{n+2}{n-2}} \to \frac{L}{V}$ in the L^2 sense, at an exponential rate.

Chapter 7

Some Results in the Scalar Positive Case

7.1 Some Facts about Stereographic Projections

Let $\vec{N} = (0, \cdots, 0, 1) = (0', 1)$ be the north pole on the unit sphere $\mathbb{S}^n$ when viewed as a subset of R^{n+1}. The ***Stereographic projection*** of $\mathbb{S}^n \setminus \{\vec{N}\}$ onto $\mathbb{R}^n$ is a bijection between $\mathbb{S}^n \setminus \{\vec{N}\}$ and $\mathbb{R}^n$ defined by the following: Consider the lines in $\mathbb{R}^{n+1}$ which go though the point $\vec{N}$. Then the point $\vec{\xi} \in \mathbb{S}^n$ is taken to the point which lies on the line determined by $\vec{\xi}$ and $\vec{N}$, and lies in the hyperplane $\{\vec{x} : x_{n+1} =< \vec{x}, \vec{N} >= 0\}$.

More explicitly we compute this map $F : \mathbb{S}^n \setminus \{\vec{N}\} \to \mathbb{R}^n$. Suppose we are given $\vec{\xi} \in \mathbb{S}^n$, we want to find the corresponding point $\vec{x} := F(\vec{\xi})$. We write

$$(x', 0) = \vec{x} = t\vec{\xi} + (1-t)\vec{N},$$

then

$$\begin{aligned}(x', 0) &= t(\xi', \xi_{n+1}) + (1-t)(0', 1)\\ &= (t\xi', t\xi_{n+1} + (1-t)).\end{aligned}$$

Therefore $t\xi_{n+1} + 1 - t = 0$, which implies $t = \frac{1}{1-\xi_{n+1}}$. Thus we have,

$$x' = \frac{\xi'}{1-\xi_{n+1}}.$$

Similarly, we can compute F^{-1}. With the same notation as above, given $\vec{x}$ we want to find $\vec{\xi}$. We write

$$\vec{\xi} = t\vec{x} + (1-t)\vec{N} = (tx', 1-t).$$

Then $\|\vec{\xi}\| = 1$ implies $\|(tx', 1-t)\| = 1$, which in turn implies $t = \frac{2}{\|x'\|^2+1}$. Therefore,

$$\vec{\xi} = (\frac{2x'}{1+\|x'\|^2}, \frac{\|x'\|^2-1}{\|x'\|^2+1}).$$

Next we give the relationship between the natural metric $g_{\mathbb{S}^n}$ [1] and $g_{\mathbb{R}^n}$.
For $\vec{\xi} = (\xi_1, \cdots, \xi_{n+1}) \in \mathbb{S}^n$ and $x' = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we have

[1] Here $g_{\mathbb{S}^n}$ is just the Euclidean metric $g_{\mathbb{R}^{n+1}}$ in $\mathbb{R}^{n+1}$ restricted to the unit sphere in $\mathbb{R}^{n+1}$.

$$
\begin{aligned}
g_{\mathbb{S}^n} &= \sum_{i=1}^{n+1} d\xi_i^2 \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{n} \frac{\partial \xi_i}{\partial x_j} dx_j)^2 + (\sum_{j=1}^{n} \frac{\partial \xi_{n+1}}{\partial x_j} dx_j)^2 \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\frac{2x_i}{1+|x'|^2}) dx_j)^2 + (\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\frac{|x'|^2-1}{|x'|^2+1}) dx_j)^2 \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{n}\{\frac{2\delta_{ij}}{1+|x'|^2} - \frac{4x_ix_j}{(1+|x'|^2)^2}\} dx_j)^2 + (\sum_{j=1}^{n} \frac{4x_j}{(1+|x'|^2)^2} dx_j)^2 \\
&= 4\sum_{i=1}^{n}(\sum_{j=1}^{n}(\frac{\delta_{ij}}{1+|x'|^2} - \frac{2x_ix_j}{(1+|x'|^2)^2}) dx_j)^2 + \frac{16}{(1+|x'|^2)^4}(\sum_{j=1}^{n} x_j dx_j)^2 \\
&= 4\sum_{i=1}^{n}\{\sum_{j=1}^{n}(\frac{\delta_{ij}}{1+|x'|^2} - \frac{2x_ix_j}{(1+|x'|^2)^2})^2 dx_j^2
\end{aligned}
$$

$$
+2\sum_{1\le j<k\le n}(\frac{\delta_{ij}}{1+|x'|^2} - \frac{2x_ix_j}{(1+|x'|^2)^2})(\frac{\delta_{ik}}{1+|x'|^2} - \frac{2x_ix_k}{(1+|x'|^2)^2}) dx_j dx_k\} + \frac{16}{(1+|x'|^2)^4}(\sum_{j=1}^{n} x_j dx_j)^2
$$

$$
= 4\sum_{i=1}^{n}\sum_{j=1}^{n}(\frac{\delta_{ij}}{(1+|x'|^2)^2} - \frac{4\delta_{ij}x_ix_j}{(1+|x'|^2)^3} + \frac{4x_i^2x_j^2}{(1+|x'|^2)^4}) dx_j^2
$$

$$
+8\sum_{i=1}^{n}\sum_{1\le j<k\le n}(\frac{\delta_{ij}\delta_{ik}}{(1+|x'|^2)^2} - \frac{2\delta_{ij}x_ix_k}{(1+|x'|^2)^3} - \frac{2\delta_{ik}x_ix_j}{(1+|x'|^2)^3} + \frac{4x_i^2x_jx_k}{(1+|x'|^2)^4}) dx_j dx_k
$$

$$
+\frac{16}{(1+|x'|^2)^4}\sum_{j=1}^{n} x_j^2 dx_j^2 + \frac{32}{(1+|x'|^2)^4}\sum_{1\le j<k\le n} x_jx_k dx_j dx_k
$$

$$
= \frac{4}{(1+|x'|^2)^2} d(x')^2 - \frac{16}{(1+|x'|^2)^3}\sum_{i=1}^{n} x_i^2 dx_i^2 + \frac{16|x'|^2}{(1+|x'|^2)^4}\sum_{i=1}^{n} x_i^2 dx_i^2 + \frac{16}{(1+|x'|^2)^4}\sum_{i=1}^{n} x_i^2 dx_i^2
$$

$$
-\frac{16}{(1+|x'|^2)^3}\sum_{i=1}^{n}\sum_{1\le i<k\le n} x_ix_k dx_i dx_k - \frac{16}{(1+|x'|^2)^3}\sum_{i=1}^{n}\sum_{1\le j<i\le n} x_ix_j dx_i dx_j
$$

$$
+\frac{32|x'|^2}{(1+|x'|^2)^4}\sum_{1\le j<k\le n} x_jx_k dx_j dx_k + \frac{32}{(1+|x'|^2)^4}\sum_{1\le j<k\le n} x_jx_k dx_j dx_k
$$

$$
= (\frac{2}{1+|x'|^2})^2 d(x')^2.
$$

Thus we have that

$$
(F^{-1})^* g_{\mathbb{S}^n} = (\frac{2}{1+|x'|^2})^2 g_{\mathbb{R}^n}. \tag{7.1}
$$

7.2 Convergence in Scalar Positive, Locally Conformally Flat Case

By the results of [15], there exists a conformal diffeomorphism $\Phi : \tilde{M} \to \Omega \subset \mathbb{S}^n$, where Ω is a dense open subset of $\mathbb{S}^n$ and $\tilde{M}$ is the universal cover of M. We first assume $(M, [g_0])$ is not conformally covered by $\mathbb{S}^n$, i.e. $\Omega \neq \mathbb{S}^n$.

Let $\tilde{g} := (\Phi^{-1})^* \pi^* g$. This is a metric defined on $\mathbb{S}^n$ and $\tilde{g} = \tilde{u}^{\frac{4}{n-2}} g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric on $\mathbb{S}^n$. We note that $\tilde{u}$ satisfies (2.1) with respect to the background metric $g_{\mathbb{S}^n}$. We get this by noting the following observation: Let $M, \tilde{M}$ be two manifolds such that $\Phi : \tilde{M} \to M$ is a diffeomorphism. Given a metric g on M, let $\tilde{g} := \Phi^* g$ and suppose g satisfies

$$g_t = (s_g - R_g) g$$

$$g|_{t=0} = g^0,$$

then given an $x \in \tilde{M}$ we have $R_{\tilde{g}}(x) = R_g(\Phi x)$. To see this, note that the diffeomorphism Φ just changes the local coordinates and $\tilde{g} = \Phi^* g$ is the pullback metric [2] of g.

Therefore, $\tilde{g}$ satisfies

$$\tilde{g}_t = (s_{\tilde{g}} - R_{\tilde{g}}) \tilde{g}$$

$$\tilde{g}|_{t=0} = \tilde{g}^0.$$

We now note the following result of [15]

Proposition 2.6 of [15] 7.2.1 *Let M be a domain in $\mathbb{S}^n$ and g a complete conformal metric on M with $|R_g|$, $|\nabla_g R_g| \leq$ Constant. Then for $\delta(x) := d(x, \partial M)$ and $g = u^{\frac{4}{n-2}} g_{\mathbb{S}^n}$, we have $u(x) \geq C\delta(x)^{-\frac{n-2}{2}}$ for $x \in M, C > 0$.*

This implies the following result

Lemma 7.2.2 *For each fixed T, $\tilde{u}(p, t) \to \infty$ uniformly for any $t \in [0, T]$ as $p \to \partial\Omega$.*

Given $p_0 \in M$, choose a point $\tilde{p}_0 \in \tilde{M}$ and a neighborhood V of $\tilde{p}_0$ such that $\pi(\tilde{p}_0) = p_0$ and $dist(\Phi(V), \partial\Omega) > 0$. Then there exists a $C > 0$ such that $\tilde{u}_0 > \frac{1}{C}$ and $\|\tilde{u}_0\|_{C^4(g_{\mathbb{S}^n})} \leq C$ on $\Phi(V)$. Note that by lemma 7.2.2, we may assume $\tilde{u}_0 > \frac{1}{C}$ holds everywhere.

Now we fix any $q_0 \in \Phi(V)$. We then view $\mathbb{S}^n$ as being embedded in $\mathbb{R}^{n+1}$ with q_0 as the North Pole, i.e. $q_0 = (0, \cdots, 0, 1)$ as an element of $\mathbb{R}^{n+1}$. Define F to be the stereographic projection and F^{-1} as its inverse. Note here that F maps the southern hemisphere onto the unit ball $\mathbb{B} \subset \mathbb{R}^n$, it maps the northern hemisphere onto $\mathbb{B}^c$ and the equator onto $\partial\mathbb{B}$.

[2] Recall that the pullback metric is given by : $\vec{\theta} \in \Omega$ and $\vec{\alpha}, \vec{\beta} \in T_{\vec{\theta}}\mathbb{S}^n$ we have

$$(\tilde{u}(\vec{\theta}))^{\frac{4}{n-2}} g_{\mathbb{S}^n}(\vec{\alpha}, \vec{\beta}) = \tilde{g}(\vec{\alpha}, \vec{\beta}) = g(d\pi(d\Phi^{-1}(\vec{\alpha})), d\pi(d\Phi^{-1}(\vec{\beta})))$$

$$= (u(\pi\Phi^{-1}\vec{\theta}))^{\frac{4}{n-2}} g_0(d\pi(d\Phi^{-1}(\vec{\alpha})), d\pi(d\Phi^{-1}(\vec{\beta}))).$$

Then for $\vec{x} \in \mathbb{R}^n$ we have

$$F^{-1}(\vec{x}) = (\frac{2\vec{x}}{1+|\vec{x}|^2}, \frac{|\vec{x}|^2-1}{|\vec{x}|^2+1}) \in \mathbb{S}^n.$$

Let [3]

$$G(\vec{x}) := F^{-1}(\frac{\vec{x}}{|\vec{x}|^2}) = (\frac{2\vec{x}}{|\vec{x}|^2+1}, \frac{1-|\vec{x}|^2}{|\vec{x}|^2+1}),$$

and note that $G(0) = q_0$.

Take any positive smooth function f defined on $\Phi(V)$, and consider the metric $f^{\frac{4}{n-2}} g_{\mathbb{S}^n}$. Let $\tilde{f}$ be given by

$$\tilde{f}^{\frac{4}{n-2}} g_{\mathbb{R}^n} = (F^{-1})^*(f^{\frac{4}{n-2}} g_{\mathbb{S}^n}),$$

where here $g_{\mathbb{R}^n}$ is the standard (Euclidean) metric on $\mathbb{R}^n$.

By Taylor's Theorem we have that

$$f \circ G(\vec{x}) = f(q_0) + \nabla(f \circ G)(0) \cdot \vec{x} + \frac{1}{2}\vec{x}^t D^2(f \circ G)(0)\vec{x} + O(|\vec{x}|^3)$$

$$= a_0 + a_i x_i + a_{ij} x_i x_j + O(|\vec{x}|^3),$$

where here $a_0 = f(q_0)$, $a_i = \frac{\partial}{\partial x_i}(f \circ G)(0)$ and $a_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}(f \circ G)(0)$. Using this, we then get an asymptotic expansion for $\tilde{f}$ near ∞.

Noting that $f \circ F^{-1}(\vec{x}) = f \circ G(\frac{\vec{x}}{|\vec{x}|^2})$ we then get that $|\vec{x}|$ large,

$$f \circ F^{-1}(\vec{x}) = a_0 + a_i \frac{x_i}{|\vec{x}|^2} + a_{ij} \frac{x_i x_j}{|\vec{x}|^4} + O(\frac{1}{|\vec{x}|^3}).$$

Also we get that

$$(F^{-1})^*(f^{\frac{4}{n-2}} g_{\mathbb{S}^n}) = (f \circ F^{-1})^{\frac{4}{n-2}} (F^{-1})^* g_{\mathbb{S}^n} = (f \circ F^{-1})^{\frac{4}{n-2}} \cdot (\frac{2}{1+|\vec{x}|^2})^2 g_{\mathbb{R}^n}.$$

Therefore,

$$\tilde{f}(\vec{x}) = (\frac{2}{1+|\vec{x}|^2})^{\frac{n-2}{2}} (f \circ F^{-1})(\vec{x}),$$

and thus

$$\tilde{f}(\vec{x}) = \frac{2^{\frac{n-2}{2}}}{(1+|\vec{x}|^2)^{\frac{n-2}{2}}} (a_0 + a_i \frac{x_i}{|\vec{x}|^2} + a_{ij} \frac{x_i x_j}{|\vec{x}|^4} + O(\frac{1}{|\vec{x}|^3}))$$

for $|\vec{x}|$ large.

The Taylor expansion about 0 for $\frac{1}{(1+t^2)^{\frac{p}{2}}}$ is $1 - \frac{p}{2}t^2 + O(t^4)$. Hence for $|t|$ large we get $\frac{1}{t^p} \frac{1}{(1+\frac{1}{t^2})^{\frac{p}{2}}}$ $= \frac{1}{t^p}(1 - \frac{p}{2t^2} + O(\frac{1}{t^4}))$. Therefore, for $|\vec{x}|$ large we get

$$\frac{1}{(1+|\vec{x}|^2)^{\frac{n-2}{2}}} = \frac{1}{|\vec{x}|^{n-2}}(1 - \frac{n-2}{2|\vec{x}|^2} + O(\frac{1}{|\vec{x}|^4})).$$

[3] G is just F^{-1} composed with an inversion on the sphere that takes $\vec{\xi} \in \mathbb{S}^n$ to $-\vec{\xi} \in \mathbb{S}^n$. Thus, G takes the unit ball $\mathbb{B}$ in $\mathbb{R}^n$ onto the northern hemisphere of $\mathbb{S}^n$, it takes $\mathbb{B}^c$ onto the southern hemisphere and $\partial\mathbb{B}$, the equator, onto itself.

Writing $\frac{1}{|\vec{x}|^2} = \frac{\delta_{ij}}{|\vec{x}|^4} x_i x_j$ we get

$$\tilde{f}(\vec{x}) = \frac{2^{\frac{n-2}{2}}}{|\vec{x}|^{n-2}}(a_0 + a_i \frac{x_i}{|\vec{x}|^2} + (a_{ij} - \frac{n-2}{2} a_0 \delta_{ij}) \frac{x_i x_j}{|\vec{x}|^4} + O(\frac{1}{|\vec{x}|^3})).$$

Differentiating with respect to x_i we have,

$$\frac{\partial}{\partial x_i}\tilde{f}(\vec{x}) = 2^{\frac{n-2}{2}}(-\frac{(n-2)}{|\vec{x}|^n} x_i (a_0 + \frac{a_j x_j}{|\vec{x}|^2}) + \frac{a_i}{|\vec{x}|^n} - \frac{2x_i}{|\vec{x}|^{n+2}} a_j x_j) + O(\frac{1}{|\vec{x}|^{n+1}}).$$

We now define w in terms of $(F^{-1})^* \tilde{g} = w^{\frac{4}{n-2}} g_{\mathbb{R}^n}$ and note that w satisfies the equation

$$\frac{\partial}{\partial t}(w^{\frac{n+2}{n-2}}) = \triangle_{g_{\mathbb{R}^n}} w + c(n) s w^{\frac{n+2}{n-2}}. \tag{7.2}$$

We now use $\tilde{u}$ as f. Thus $a_0 = a_0(t) = \tilde{u}(q_0, t)$, $a_i = a_i(t) = \frac{\partial(\tilde{u}(\cdot,t)\circ G)}{\partial x_i}(0)$, $a_{ij} = a_{ij}(t) = \frac{\partial^2(\tilde{u}(\cdot,t)\circ G)}{\partial x_i \partial x_j}(0)$. Let $\vec{y}(t) = (y_1(t), \cdots, y_n(t))$ where $y_i(t) = -\frac{a_i(t)}{(n-2)a_0(t)}$; we call $\vec{y}(t)$ the ***center of*** $w(\cdot, t)$.[4]

7.2.1 Some results from [4]

Let $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{y} = (y_1, \cdots, y_n)$. Then for positive q we have [5]

$$\frac{1}{|\vec{x} - \vec{y}|^q} = \frac{1}{|\vec{x}|^q}(1 + \frac{q}{|\vec{x}|^2} x_i y_i + \cdots).$$

Assume that we have a function v which has the following asymptotic expansion for large $|\vec{x}|$, where we will assume that $a_0 > 0$,

$$v(\vec{x}) = \frac{1}{|\vec{x}|^m}(a_0 + \frac{a_j x_j}{|\vec{x}|^2} + \frac{a_{jk} x_j x_k}{|\vec{x}|^4} + o(\frac{1}{|\vec{x}|^2})),$$

and

$$v_{x_i}(\vec{x}) = \frac{-m}{|\vec{x}|^{m+2}} x_i (a_0 + \frac{a_j x_j}{|\vec{x}|^2}) + \frac{a_i}{|\vec{x}|^{m+2}} - \frac{2x_i}{|\vec{x}|^{m+4}} a_j x_j + O(\frac{1}{|\vec{x}|^{m+3}}). \tag{7.3}$$

We then define $\vec{y} = (y_1, \cdots, y_n)$, where $y_j = \frac{-a_j}{m a_0}$. By the above discussion, and by replacing $\vec{x}$ by $\vec{x} - \vec{y}$ in 7.3 we have

[4] The expansion given above holds for $w(\cdot, t)$ with $a_0 = a_0(t)$, $a_i = a_i(t)$ and $a_{ij} = a_{ij}(t)$, and the expansion is uniform $\forall t \in [0, T]$ where $T \in [0, T^*)$, where we are assuming our solution to (1.2) exists on $M \times [0, T^*)$.

[5] To establish this we define $f(\vec{x}) := \frac{1}{|\vec{x} - \vec{y}|^q}$ and we examine the Taylor expansion of f about the origin. We have that

$$f_{x_i} = -\frac{q}{|\vec{x} - \vec{y}|^{q+2}}(x_i - y_i),$$

thus

$$\begin{aligned} f(\vec{x}) &= f(0) + \nabla f(0) \cdot \vec{x} + \cdots \\ &= \frac{1}{|\vec{y}|^q} + \frac{q}{|\vec{y}|^{q+2}} \vec{y} \cdot \vec{x} + \cdots \\ &= \frac{1}{|\vec{y}|^q}(1 + \frac{q}{|\vec{y}|^2} x_i y_i + \cdots). \end{aligned}$$

Thus we get our desired result.

$$\begin{aligned}
v(\vec{x}-\vec{y}) &= \frac{1}{|\vec{x}|^m}(1+\frac{m}{|\vec{x}|^2}x_iy_i+\cdots)(a_o+\frac{a_j(x_j-y_j)}{|\vec{x}-\vec{y}|^2}+\frac{a_{jk}(x_j-y_j)(x_k-y_k)}{|\vec{x}-\vec{y}|^4}+o(\frac{1}{|\vec{x}|^2}))\\
&= \frac{1}{|\vec{x}|^m}(1-\frac{a_ix_i}{a_0|\vec{x}|^2}+\cdots)(a_o+a_j(x_j+\frac{a_j}{ma_0})\cdot\frac{1}{|\vec{x}|^2}\cdot(1+\frac{2x_iy_i}{|\vec{x}|^2}+\cdots)\\
&\quad +a_{jk}(x_j-y_j)(x_k-y_k)\cdot\frac{1}{|\vec{x}|^4}\cdot(1+\frac{4x_iy_i}{|\vec{x}|^2}+\cdots)+o(\frac{1}{|\vec{x}|^2}))\\
&= \frac{1}{|\vec{x}|^m}(a_0+\frac{a_jx_j}{|\vec{x}|^2}+\frac{a_{jk}x_jx_k}{|\vec{x}|^4}+o(\frac{1}{|\vec{x}|^2})-\frac{a_ix_i}{a_0|\vec{x}|^2}\{a_0+\frac{a_jx_j}{|\vec{x}|^2}+\frac{a_{jk}x_jx_k}{|\vec{x}|^4}+o(\frac{1}{|\vec{x}|^2})\})\\
&= \frac{1}{|\vec{x}|^m}(a_0+\frac{a_{jk}x_jx_k}{|\vec{x}|^4}-\frac{1}{a_0|\vec{x}|^4}(a_ix_i)(a_jx_j)+o(\frac{1}{|\vec{x}|^2}))\\
&= \frac{1}{|\vec{x}|^m}(a_0+\frac{\tilde{a}_{jk}x_jx_k}{|\vec{x}|^4}+o(\frac{1}{|\vec{x}|^2})).
\end{aligned}$$

Likewise,

$$\begin{aligned}
v_{x_i}(\vec{x}-\vec{y}) &= -\frac{m(x_i-y_i)}{|\vec{x}|^{m+2}}(1+(m+2)\frac{x_ky_k}{|\vec{x}|^2}+\cdots)\{a_0+\frac{a_j(x_j-y_j)}{|\vec{x}|^2}(1+\frac{2x_ky_k}{|\vec{x}|^2}+\cdots)\}\\
&\quad +\frac{a_i}{|\vec{x}|^{m+2}}(1+\frac{(m+2)x_ky_k}{|\vec{x}|^2}+\cdots)-\frac{2(x_i-y_i)a_j(x_j-y_j)}{|\vec{x}|^{m+4}}(1+\frac{(m+4)x_ky_k}{|\vec{x}|^2}+\cdots)+O(\frac{1}{|\vec{x}|^{m+3}})\\
&= -\frac{m(x_i-y_i)}{|\vec{x}|^{m+2}}(1+(m+2)\frac{x_ky_k}{|\vec{x}|^2}+\cdots)\{a_0+\frac{a_jx_j}{|\vec{x}|^2}+O(\frac{1}{|\vec{x}|})\}+\frac{a_i}{|\vec{x}|^{m+2}}(1+O(\frac{1}{|\vec{x}|}))\\
&\quad -\frac{2(x_i-y_i)a_j(x_j-y_j)}{|\vec{x}|^{m+4}}(1+O(\frac{1}{|\vec{x}|}))+O(\frac{1}{|\vec{x}|^{m+3}})\\
&= -\frac{m(x_i-y_i)}{|\vec{x}|^{m+2}}(1+(m+2)\frac{x_ky_k}{|\vec{x}|^2}+\cdots)\{a_0+\frac{a_jx_j}{|\vec{x}|^2}\}+\frac{a_i}{|\vec{x}|^{m+2}}-\frac{2x_ia_jx_j}{|\vec{x}|^{m+4}}+O(\frac{1}{|\vec{x}|^{m+3}})\\
&= \frac{-m(x_i+\frac{a_i}{ma_0})a_0}{|\vec{x}|^{m+2}}-\frac{m(m+2)(x_i-y_i)}{|\vec{x}|^{m+4}}x_ky_ka_0-\frac{m(x_i-y_i)a_jx_j}{|\vec{x}|^{m+4}}+\frac{a_i}{|\vec{x}|^{m+2}}-\frac{2x_ia_jx_j}{|\vec{x}|^{m+4}}+O(\frac{1}{|\vec{x}|^{m+2}})\\
&= \frac{-mx_ia_0}{|\vec{x}|^{m+2}}-\frac{a_i}{|\vec{x}|^{m+2}}-\frac{m(m+2)x_i}{|\vec{x}|^{m+4}}x_ky_ka_0-\frac{mx_ia_jx_j}{|\vec{x}|^{m+4}}+\frac{a_i}{|\vec{x}|^{m+2}}-\frac{2x_ia_jx_j}{|\vec{x}|^{m+4}}+O(\frac{1}{|\vec{x}|^{m+2}})\\
&= \frac{-mx_ia_0}{|\vec{x}|^{m+2}}-\frac{(m+2)x_i}{|\vec{x}|^{m+4}}x_ka_k-\frac{mx_ia_jx_j}{|\vec{x}|^{m+4}}-\frac{2x_ia_jx_j}{|\vec{x}|^{m+4}}+O(\frac{1}{|\vec{x}|^{m+2}})\\
&= \frac{-mx_ia_0}{|\vec{x}|^{m+2}}+O(\frac{1}{|\vec{x}|^{m+3}}).
\end{aligned}$$

Thus, for a suitable C_0, R_1, if $x_1 \geq \frac{C_0}{|\vec{x}|}$ and $|\vec{x}| \geq R_1$ we get $v_{x_1} < 0$. To see this, we require that $-ma_0x_1 + O(\frac{1}{|\vec{x}|}) < 0$. Note that we have $-ma_0x_1 + O(\frac{1}{|\vec{x}|}) \leq -ma_0x_1 + C\frac{1}{|\vec{x}|}$ for some positive C. Assuming that $-ma_0x_1 + \frac{C}{|\vec{x}|} < 0$, we get $-ma_0x_1 < -\frac{C}{|\vec{x}|}$, or equivalently $x_1 > \frac{C}{ma_0|\vec{x}|}$.

In the following discussion we let $\tilde{v}(x) := v(x-y)$.

Lemma 4.1 of [4] 7.2.3 *For $\lambda > 0$, there exists $R = R(\lambda)$ depending only on* $\min\{1,\lambda\}$ *(as well as on v) such that for $\vec{x} = (x_1, x')$, $\vec{z} = (z_1, x')$ satisfying $x_1 < z_1$, $x_1 + z_1 \geq 2\lambda$, $|\vec{x}| \geq R$, we have $\tilde{v}(\vec{x}) > \tilde{v}(\vec{z})$.*

Proof: We shall show that if we have a pair of points $\vec{x} = (x_1, x')$, $\vec{z} = (z_1, x')$ with $|\vec{x}| \geq R_1$, $z_1 > x_1$, $z_1 + x_1 \geq 2\lambda$ for which $\tilde{v}(\vec{x}) \leq \tilde{v}(\vec{z})$, then necessarily $|\vec{x}|, |\vec{z}| \leq R = R(\min\{1, \lambda\})$. Note that $|\vec{z}| > |\vec{x}|$ holds under these assumptions. By our assumption that $\tilde{v}(\vec{x}) \leq \tilde{v}(\vec{z})$ we then get

$$0 \geq \tilde{v}(\vec{x}) - \tilde{v}(\vec{z}) = \frac{1}{|\vec{x}|^m}(a_0 + \tilde{a}_{ij}\frac{x_i x_j}{|\vec{x}|^4} + o(\frac{1}{|\vec{x}|^2})) - \frac{1}{|\vec{z}|^m}(a_0 + \tilde{a}_{ij}\frac{z_i z_j}{|\vec{z}|^4} + o(\frac{1}{|\vec{z}|^2})),$$

which implies

$$\begin{aligned} a_0(\frac{1}{|\vec{x}|^m} - \frac{1}{|\vec{z}|^m}) &\leq \tilde{a}_{ij}(\frac{z_i z_j}{|\vec{z}|^{m+4}} - \frac{x_i x_j}{|\vec{x}|^{m+4}}) + o(\frac{1}{|\vec{z}|^{m+2}}) + o(\frac{1}{|\vec{x}|^{m+2}}) \\ &\leq \tilde{a}_{ij}(\frac{z_i z_j}{|\vec{z}|^{m+4}} - \frac{x_i x_j}{|\vec{x}|^{m+4}}) + C_1(\frac{1}{|\vec{x}|^{m+3}} + \frac{1}{|\vec{z}|^{m+3}}) \\ &\leq \frac{C}{|\vec{x}|^{m+2}}. \end{aligned}$$

Since $|\vec{z}| > |\vec{x}|$, for $p > 1$ we have $|\vec{z}|^{p-1} \geq |\vec{x}|^{p-1}$ and thus $\frac{1}{|\vec{z}|} \geq \frac{|\vec{x}|^{p-1}}{|\vec{z}|^p}$. This implies $\frac{1}{|\vec{x}|} - \frac{|\vec{x}|^{p-1}}{|\vec{z}|^p} \geq \frac{1}{|\vec{x}|} - \frac{1}{|\vec{z}|}$. Thus, $\frac{1}{|\vec{x}|^p} - \frac{1}{|\vec{z}|^p} \geq \frac{1}{|\vec{x}|^{p-1}}(\frac{1}{|\vec{x}|} - \frac{1}{|\vec{z}|})$.

Therefore, we get

$$\frac{1}{|\vec{x}|^{m-1}}(\frac{1}{|\vec{x}|} - \frac{1}{|\vec{z}|}) \leq \frac{1}{|\vec{x}|^m} - \frac{1}{|\vec{z}|^m} \leq \frac{\tilde{C}}{|\vec{x}|^{m+2}}.$$

This implies

$$\frac{|\vec{z}|}{|\vec{x}||\vec{z}|} - \frac{|\vec{x}|}{|\vec{x}||\vec{z}|} = \frac{1}{|\vec{x}|} - \frac{1}{|\vec{z}|} \leq \frac{\tilde{C}}{|\vec{x}|^3},$$

and thus

$$|\vec{z}| - |\vec{x}| \leq \frac{\tilde{C}|\vec{z}|}{|\vec{x}|^2} = \tilde{C}(\frac{|\vec{z}| - |\vec{x}|}{|\vec{x}|^2} + \frac{1}{|\vec{x}|}).$$

Hence if $\frac{\tilde{C}}{|\vec{x}|^2} \leq \frac{1}{2}$ we have

$$|\vec{z}| - |\vec{x}| \leq \tilde{C}(\frac{|\vec{z}| - |\vec{x}|}{|\vec{x}|^2} + \frac{1}{|\vec{x}|}) \leq \frac{1}{2}(|\vec{z}| - |\vec{x}|) + \frac{\tilde{C}}{|\vec{x}|},$$

which implies

$$|\vec{z}| - |\vec{x}| \leq \frac{2\tilde{C}}{|\vec{x}|}.$$

Thus

$$|\vec{z}| \leq \frac{2\tilde{C}}{|\vec{x}|} + |\vec{x}| = 2|\vec{x}| \cdot \frac{\tilde{C}}{|\vec{x}|^2} + |\vec{x}| \leq 2|\vec{x}|(\frac{1}{2}) + |\vec{x}| = 2|\vec{x}|.$$

Therefore, if $|\vec{x}|$ is large enough, and the desired inequality doesn't hold, then we must have $|\vec{z}| \leq 2|\vec{x}|$.

We just assumed that $|\vec{x}|^2 \geq 2\tilde{C}$ to arrive at the last inequality. However, we may assume this from now on. To see this, suppose $|\vec{x}|^2 \leq 2\tilde{C}$. Because $\tilde{v}(\vec{z}) \to 0$ as $|\vec{z}| \to \infty$ and we are assuming $\tilde{v}(\vec{x}) < \tilde{v}(\vec{z})$, we then get that $|\vec{z}| \leq R$, for some R independent of λ.

Once again,

$$a_0(\frac{1}{|\vec{x}|^m}-\frac{1}{|\vec{z}|^m}) \leq \tilde{a}_{ij}(\frac{z_iz_j}{|\vec{z}|^{m+4}}-\frac{x_ix_j}{|\vec{x}|^{m+4}})+C_1(\frac{1}{|\vec{x}|^{m+3}}+\frac{1}{|\vec{z}|^{m+3}})$$

$$=\tilde{a}_{11}(\frac{z_1^2}{|\vec{z}|^{m+4}}-\frac{x_1^2}{|\vec{x}|^{m+4}})+2\sum_{j>1}\tilde{a}_{1j}(\frac{z_1z_j}{|\vec{z}|^{m+4}}-\frac{x_1x_j}{|\vec{x}|^{m+4}})+\sum_{j,k>1}\tilde{a}_{jk}(\frac{z_jz_k}{|\vec{z}|^{m+4}}-\frac{x_jx_k}{|\vec{x}|^{m+4}})+C_1(\frac{1}{|\vec{x}|^{m+3}}+\frac{1}{|\vec{z}|^{m+3}})$$

$$=\tilde{a}_{11}(\frac{z_1^2}{|\vec{z}|^{m+4}}-\frac{x_1^2}{|\vec{x}|^{m+4}})+2\sum_{j>1}\tilde{a}_{1j}x_j(\frac{z_1}{|\vec{z}|^{m+4}}-\frac{x_1}{|\vec{x}|^{m+4}})$$

$$+\sum_{j,k>1}\tilde{a}_{jk}x_jx_k(\frac{1}{|\vec{z}|^{m+4}}-\frac{1}{|\vec{x}|^{m+4}})+C_1(\frac{1}{|\vec{x}|^{m+3}}+\frac{1}{|\vec{z}|^{m+3}}).$$

We then see that

$$\frac{1}{|\vec{x}|^{m-1}}(\frac{1}{|\vec{x}|}-\frac{1}{|\vec{z}|}) \leq \frac{1}{|\vec{x}|^m}-\frac{1}{|\vec{z}|^m} \leq \frac{\tilde{a}_{11}}{a_0}(\frac{z_1^2}{|\vec{z}|^{m+4}}-\frac{x_1^2}{|\vec{x}|^{m+4}})$$

$$+2\sum_{j>1}\frac{\tilde{a}_{1j}}{a_0}x_j(\frac{z_1}{|\vec{z}|^{m+4}}-\frac{x_1}{|\vec{x}|^{m+4}})+\sum_{j,k>1}\frac{\tilde{a}_{jk}}{a_0}x_jx_k(\frac{1}{|\vec{z}|^{m+4}}-\frac{1}{|\vec{x}|^{m+4}})+\frac{C_1}{a_0}(\frac{1}{|\vec{x}|^{m+3}}+\frac{1}{|\vec{z}|^{m+3}})$$

$$\leq C\frac{z_1^2-x_1^2}{|\vec{x}|^{m+4}}+C\frac{z_1-x_1}{|\vec{x}|^{m+3}}+C|\vec{x}|^2(\frac{1}{|\vec{x}|^{m+4}}-\frac{1}{|\vec{z}|^{m+4}})+C\frac{1}{|\vec{x}|^{m+3}}$$ [6]

$$\leq C\frac{z_1^2-x_1^2}{|\vec{z}|^{m+4}}+C\frac{z_1-x_1}{|\vec{x}|^{m+3}}+\frac{C}{|\vec{x}|^{m+1}}(\frac{1}{|\vec{x}|}-\frac{1}{|\vec{z}|})+\frac{C}{|\vec{x}|^{m+3}}.$$

This implies,

$$\frac{|\vec{z}|-|\vec{x}|}{|\vec{x}||\vec{z}|}=\frac{1}{|\vec{x}|}-\frac{1}{|\vec{z}|} \leq C\frac{z_1^2-x_1^2}{|\vec{x}|^5}+\frac{C}{|\vec{x}|^2}(\frac{1}{|\vec{x}|}-\frac{1}{|\vec{z}|})+C\frac{z_1-x_1}{|\vec{x}|^4}+\frac{C}{|\vec{x}|^4}.$$

Therefore [7]

$$|\vec{z}|-|\vec{x}| \leq C\frac{|\vec{z}|}{|\vec{x}|^4}(|\vec{z}|^2-|\vec{x}|^2)+C\frac{|\vec{z}|}{|\vec{x}|}(\frac{1}{|\vec{x}|}-\frac{1}{|\vec{z}|})+C\frac{|\vec{z}|}{|\vec{x}|^3}(z_1-x_1)+C\frac{|\vec{z}|}{|\vec{x}|^3}$$

$$\leq 2C\frac{1}{|\vec{x}|^3}(|\vec{z}|^2-|\vec{x}|^2)+2C(\frac{1}{|\vec{x}|}-\frac{1}{|\vec{z}|})+2C\frac{1}{|\vec{x}|^2}(z_1-x_1)+2C\frac{1}{|\vec{x}|^2}$$

$$\leq 2C\frac{1}{|\vec{x}|^3}(|\vec{z}|^2-|\vec{x}|^2)+2C\frac{1}{|\vec{x}|^2}(|\vec{z}|-|\vec{x}|)+2C\frac{1}{|\vec{x}|^2}(z_1-x_1)+2C\frac{1}{|\vec{x}|^2}.$$

Multiplying by $|\vec{z}|+|\vec{x}|$ we get

$$|\vec{z}|^2-|\vec{x}|^2=z_1^2-x_1^2$$

$$\leq \frac{2C}{|\vec{x}|^3}(|\vec{z}|+|\vec{x}|)(|\vec{z}|^2-|\vec{x}|^2)+\frac{2C}{|\vec{x}|^2}(z_1^2-x_1^2)+\frac{2C}{|\vec{x}|^2}(|\vec{z}|+|\vec{x}|)(z_1-x_1)+\frac{2C}{|\vec{x}|^2}(|\vec{z}|+|\vec{x}|)$$

$$\leq {}^{8}\,\frac{6C}{|\vec{x}|^2}(z_1^2-x_1^2)+\frac{2C}{|\vec{x}|^2}(z_1^2-x_1^2)+\frac{6C}{|\vec{x}|}(z_1-x_1)+\frac{6C}{|\vec{x}|}$$

[6] Note that

$$|\vec{x}|^2(\frac{1}{|\vec{x}|^{m+4}}-\frac{1}{|\vec{z}|^{m+4}}) \leq \frac{1}{|\vec{x}|^{m+1}}(\frac{1}{|\vec{x}|}-\frac{1}{|\vec{z}|}).$$

[7] Use $z_1^2-x_1^2=|\vec{z}|^2-|\vec{x}|^2$ and $|\vec{z}| \leq 2|\vec{x}|$

[8] Use $|\vec{x}|+|\vec{z}| \leq 3|\vec{x}|$

$$= \frac{8C}{|\vec{x}|^2}(z_1^2 - x_1^2) + \frac{6C}{|\vec{x}|}(z_1 - x_1) + \frac{6C}{|\vec{x}|}$$
$$\leq^{9} \frac{4C}{\tilde{C}}(z_1^2 - x_1^2) + \frac{6C}{|\vec{x}|}(z_1 - x_1) + \frac{6C}{|\vec{x}|}.$$

But before, $\tilde{C}$ was only chosen such that $\frac{1}{|\vec{x}|^m} - \frac{1}{|\vec{z}|^m} \leq \frac{\tilde{C}}{|\vec{x}|^{m+2}}$. Hence we may make $\tilde{C}$ larger if necessary so we may assume that $\frac{4C}{\tilde{C}} \leq \frac{1}{2}$. Thus,

$$z_1^2 - x_1^2 \leq \frac{12C}{|\vec{x}|}(z_1 - x_1) + \frac{12C}{|\vec{x}|}.$$

Also, $z_1^2 - x_1^2 \geq 2\lambda(z_1 - x_1)$, which implies

$$2\lambda(z_1 - x_1) \leq \frac{12C}{|\vec{x}|}(z_1 - x_1) + \frac{12C}{|\vec{x}|},$$

or equivalently,

$$(2\lambda - \frac{12C}{|\vec{x}|})(z_1 - x_1) \leq \frac{12C}{|\vec{x}|}.$$

Thus if $2\lambda - \frac{12C}{|\vec{x}|} \geq \lambda$ [10] we get

$$\lambda(z_1 - x_1) \leq (2\lambda - \frac{12C}{|\vec{x}|})(z_1 - x_1) \leq \frac{12C}{|\vec{x}|},$$

which implies

$$z_1 - x_1 \leq \frac{12C}{\lambda|\vec{x}|}.$$

But then we have $z_1 \geq 2\lambda - x_1$. Therefore $2\lambda - 2x_1 \leq z_1 - x_1 \leq \frac{12C}{2\lambda|\vec{x}|}$, or equivalently, $x_1 \geq \lambda - \frac{12C}{2\lambda|\vec{x}|}$.

But then, $\lambda - \frac{12C}{2\lambda|\vec{x}|} \geq \frac{C_0}{|\vec{x}|}$, provided we assume $|\vec{x}| \geq \frac{12C}{2\lambda^2} + \frac{C_0}{\lambda}$.

Hence we can conclude that $x_1 \geq \frac{C_0}{|\vec{x}|}$, which implies that $v_{x_1} < 0$. Thus v is strictly decreasing on the line segment from $\vec{x}$ to $\vec{z}$, which contradicts the assumption that $\tilde{v}(\vec{x}) \leq \tilde{v}(\vec{z})$.
Q.E.D.

For $\vec{x} = (x_1, x')$ we let $\vec{x}^\lambda := (2\lambda - x_1, x')$ be the reflection of $\vec{x}$ about the plane $\{x_1 = \lambda\}$.

Lemma 4.2 of [4] 7.2.4 *There exists $\lambda_0 \geq 0$ such that for all $\lambda \geq \lambda_0$,*

$$\tilde{v}(\vec{x}) > \tilde{v}(\vec{x}^\lambda) \qquad \text{if} \quad x_1 < \lambda. \tag{7.4}$$

Proof: Let $R_1 = \max\{1, R(1)$ from Lemma 7.2.3$\}$. By lemma 7.2.3, if $|\vec{x}| > R_1$, $\lambda \geq 1$ and $x_1 < \lambda$ we have

$$\tilde{v}(\vec{x}) > \tilde{v}(\vec{x}^\lambda). \tag{7.5}$$

But $\tilde{v}(\vec{x}) \geq c_0$ for $|\vec{x}| \leq R_1$[11]. Furthermore, for $R_2 > 1$ sufficiently large, we have $v(\vec{z}) < c_0$ for $|\vec{z}| \geq R_2$. Thus (7.4) holds if $\lambda \geq R_2$ and $|\vec{x}| \leq R_1$. Combining this with (7.5) we obtain (7.4) with $\lambda_0 = R_2$.
Q.E.D.

[9] Use $\frac{1}{|\vec{x}|^2} \leq \frac{1}{2\tilde{C}}$
[10] i.e. $|\vec{x}| \geq \frac{12C}{\lambda}$
[11] We are assuming v is positive.

7.2.2 Convergence in the Scalar Positive, Locally Conformally Flat Case

Theorem 7.2.5 *There exists a* $\tilde{C} > 0$ *depending on* $\mathrm{diam}(\partial\Omega)$, $\mathrm{dist}(q_0, \partial\Omega)$, $\mathrm{dist}(q_0, \partial\Phi(V))$ *and the constant* C *such that* $\tilde{u}_0 > C^{-1}$, $\|\tilde{u}_0\|_{C^4} \leq C$ *on* $\Phi(V)$, *so that we have* $|\vec{y}(t)| \leq \tilde{C}$ *for all* $t \in [0, T^*)$.

Proof: Fix $T \in (0, T^*)$. By possibly performing a rotation of coordinates and the transformation $x_n \mapsto -x_n$, we may assume that $y_n(T) = \max_{1 \leq i \leq n} |y_i(T)|$.[12]

By the expansion for $w(\cdot, t)$ and the arguments of lemma 7.2.4 we get that there exists a $\lambda_0 \geq 1$ such that for all $\lambda \geq \lambda_0$ $w_0(\vec{x}) > w_0(\vec{x}^\lambda)$ whenever $x_n < \lambda$, where here $w_0(\vec{x}) := w(\vec{x}, 0)$, $\vec{x} = (x', x_n)$ and $\vec{x}^\lambda := (x', 2\lambda - x_n)$.

However, we note that w_0 is only defined on $\mathbb{R}^n \setminus F(\partial\Omega)$, but the results of [4] still apply here by way of the lemma 7.2.2 which is a corollary of proposition 7.2.1 from [15]. Also, we note here that λ_0 can still be estimated in terms of the constant C such that $\tilde{u}_0 > \frac{1}{C}$ and $\|\tilde{u}_0\|_{C^4(g_{\mathbb{S}^n})} \leq C$ on $\Phi(V)$, $\mathrm{dist}(q_0, \partial\Omega)$, $\mathrm{diam}(\partial\Omega)$ and $\mathrm{dist}(q_0, \partial\Phi(V))$.

We will assume that $F(\partial\Omega)$ lies strictly below the plane $\{\vec{x} : x_n = \lambda_0\}$. This is possible since there is a neighborhood $\tilde{N}$ of the north pole on $\mathbb{S}^n$ so that $\tilde{N} \cap \partial\Omega = \phi$, thus $F(\partial\Omega)$ must stay in some bounded set. Then we claim that $y_n(T) \leq \lambda_0$. By the arguments from [4] and the fact that the expansion for $w(\cdot, t)$ is uniform for all $t \in [0, T]$ we get that there exists a $\lambda_1 \geq \lambda_0$ such that for all $\lambda \geq \lambda_1$, $w(\vec{x}, t) > w(\vec{x}^\lambda, t)$ whenever $t \in [0, T]$ and $x_n < \lambda$.

Now we begin the procedure of moving the plane $\{\vec{x} : x_n = \lambda\}$ by decreasing λ. Let $w^\lambda(\vec{x}, t) := w(\vec{x}^\lambda, t)$. Then w^λ satisfies

$$\frac{\partial}{\partial t}(w^\lambda)^{\frac{n+2}{n-2}} = \triangle_{\mathbb{R}^n} w^\lambda + c(n) s (w^\lambda)^{\frac{n+2}{n-2}}$$

and $w^\lambda \equiv w$ when restricted to the plane $\{\vec{x} : x_n = \lambda\}$. We then restrict w^λ to the set $\{(\vec{x}, t) : x_n \leq \lambda, \vec{x} \notin F(\partial\Omega), 0 \leq t \leq T\}$ and define $I := \{\lambda : \lambda > \lambda_0, \lambda > \max_{0 \leq t \leq T} y_n(t), w^\lambda \leq w\}$.

Because $w(\vec{x}, t) > w(\vec{x}^\lambda, t)$ whenever $t \in [0, T]$ and $x_n < \lambda$, we have I is nonempty. Also, $w^\lambda \equiv w$ can never happen for $\lambda \geq \lambda_0$ because $w_0(\vec{x}) > w_0(\vec{x}^\lambda)$ whenever $\lambda \geq \lambda_0$ and $x_n < \lambda$.[13] Our goal is to show that $I = (\lambda_0, \infty)$ which implies $y(T) \leq \lambda_0$; thus implying our result.

Thus given any $\lambda \in I$; we consider the function $v := w - w^\lambda$. Then v satisfies the following

$$v_t = \frac{n-1}{w^{\frac{4}{n-2}}} \triangle v + bv,$$

$$v(\cdot, 0)|_{\{x_n = \lambda\}} = 0,$$

$$v(\cdot, 0)|_{\{x_n < \lambda\}} > 0,$$

[12] This assumption is allowed since the equation

$$w_t = \triangle w + csw^{\frac{n+2}{n-2}}$$

is invariant under such transformations, namely because $\triangle$ is.

[13] Also, one may use the singular set $F(\partial\Omega)$ to rule out $w^\lambda \equiv w$.

where $b := (n-1)c(n)s(t) - \frac{1}{c(n)}\triangle w^{\lambda}\int_0^1 \frac{1}{(\tau w+(1-\tau)w^{\lambda})^N}d\tau$.
Then a maximum principle argument implies that $v > 0$ for $x_n < \lambda$, or simply $w > w^{\lambda}$ for $x_n < \lambda$. Moreover, the Hopf boundary point lemma implies that $\frac{\partial w}{\partial x_n}|_{\{x_n=\lambda\}} < 0$.

For each fixed $t \in [0, T]$ we expand $w(\cdot, t)$ around the point $\vec{y}(t)$ to get the following [14]

$$w(\vec{x} - \vec{y}(t), t) = \frac{2^{\frac{n-2}{2}}}{|\vec{x}|^{n-2}}(a_0 + \tilde{a}_{ij}\frac{x_i x_j}{|\vec{x}|^4} + O(\frac{1}{|\vec{x}|^3})),$$

$$\frac{\partial w}{\partial x_i}(\vec{x} - \vec{y}(t), t) = -\frac{(n-2)2^{\frac{n-2}{2}}}{|\vec{x}|^n}a_0 x_i + O(\frac{1}{|\vec{x}|^{n+1}}),$$

where $(\tilde{a}_{ij})$ are as specified above in the discussion on the results of [4].

The plane $\{x_n = \lambda\} \mapsto \{x_n = \lambda - y_n(t)\}$, where now we view $\vec{y}(t)$ as the origin. Because $\lambda \in I$, we have $\lambda - y_n(t) > 0$.

Next we'll show that I is open in (λ_0, ∞). To do this we essentially use Lemma 4.3 and Lemma 4.4 from [4].

Lemma 7.2.6 *For t fixed, assume that for some $\lambda > 0$ we have $w(\vec{x}, t) \geq w(\vec{x}^{\lambda}, t)$, $w(\cdot, t) \neq w^{\lambda}(\cdot, , t)$ as functions for $x_n < \lambda$, then $w(\vec{x}, t) > w(\vec{x}^{\lambda}, t)$ if $x_n < \lambda$ and $w_{x_n}(\cdot, t)|_{\{x_n=\lambda\}} < 0$.*

Proof: We define $v := w^{\lambda} - w$, which satisfies $v \leq 0$ and v is not identically equal to 0. v satisfies a linear parabolic equation, and thus the maximum principle and Hopf boundary lemma yield the result.
Q.E.D.

Lemma 7.2.7 *The set of λ for which*

$$w(\vec{x}, t) > w(\vec{x}^{\lambda}, t) \quad \text{if} \quad x_n < \lambda \tag{7.6}$$

is open.

Proof: Suppose (7.6) holds for $\lambda = \bar{\lambda} > 0$. Set $\bar{R} := R(\frac{\bar{\lambda}}{2})$ from lemma 7.2.3. Then (7.6) holds for $\lambda \geq \frac{\bar{\lambda}}{2}$ provided $|\vec{x}| > \bar{R}$. Thus we only have to consider when $|\vec{x}| \leq \bar{R}$. If (7.6) did not hold for all λ in some neighborhood of $\bar{\lambda}$, then there would be a sequence $\{\vec{x}_j\}_{j=1}^{\infty}$ with $|\vec{x}_j| \leq \bar{R}$, and a sequence $\lambda_j \to \bar{\lambda}$, $\lambda_j \geq \frac{\bar{\lambda}}{2}$ with $(x_j)_n \leq \lambda_j$ and $w(\vec{x}_j, t) \leq w(\vec{x}_j^{\lambda_j}, t)$. Then there exists a subsequence $\{\vec{x}_{j_k}\}_{k=1}^{\infty}$ such that $\vec{x}_{j_k} \to \tilde{x}$, $|\tilde{x}| \leq \bar{R}$, where $w(\tilde{x}, t) \leq w(\tilde{x}^{\bar{\lambda}}, t)$. But by (7.6) we must have $\tilde{x}_n = \bar{\lambda}$. Also, we must have $w_{x_n}(\tilde{x}, t) \geq 0$, which is a contradiction to the previous lemma 7.2.2.
Q.E.D.

Therefore, given any $\lambda \in I$, there exists a $\varepsilon(t) > 0$ such that if $\lambda' \in (\lambda - \varepsilon(t), \lambda + \varepsilon(t))$, then $w^{\lambda'}(\cdot, t) \leq w(\cdot, t)$. Since $\lambda > \max_{0 \leq t \leq T} y_n(t)$, and that our asymptotic expansion for $w(\cdot, t)$ [15] is

[14] See the above discussion on the results of [4] to see the explicit calculations involved in verifying this.
[15] Here we view $\vec{y}(t)$ as the center of the asymptotic expansion.

uniform for all $t \in [0,T]$, we can choose $\varepsilon(t)$ uniformly for all $t \in [0,T]$. Thus there exists $\varepsilon > 0$ such that $(\lambda-\varepsilon, \lambda+\varepsilon) \subset I$ which implies that I is open in (λ_0, ∞).

Next we will show that I is closed in (λ_0, ∞).

Let $\lambda \in \bar{I}$. By continuity, we have $w^\lambda \leq w$ and $\lambda \geq \max_{0\leq t\leq T} y_n(t)$. If $\lambda = \max_{0\leq t\leq T} y_n(t)$, then there exists $t_0 \in [0,T]$ such that $\lambda = y_n(t_0)$. Now let $\vec{y}(t_0)$ be the origin. [16] We denote $F(\partial\Omega)$ by $\tilde{\Gamma}$, and consider the stereographic projection $F : \mathbb{S}^n \to \mathbb{R}^n$. We define z and z^λ by

$$F^*(w^{\frac{4}{n-2}} g_{\mathbb{R}^n}) = z^{\frac{4}{n-2}} g_{\mathbb{S}^n}$$

and

$$F^*((w^\lambda)^{\frac{4}{n-2}} g_{\mathbb{R}^n}) = (z^\lambda)^{\frac{4}{n-2}} g_{\mathbb{S}^n}.$$

For $\vec{x} \in \mathbb{S}^n$, $\vec{\alpha}, \vec{\beta} \in T_{\vec{x}}\mathbb{S}^n$ we have

$$F^*(w^{\frac{4}{n-2}} g_{\mathbb{R}^n})(\vec{\alpha}, \vec{\beta}) = w^{\frac{4}{n-2}}(F(\vec{x})) g_{\mathbb{R}^n}(dF\vec{\alpha}, dF\vec{\beta}) = w^{\frac{4}{n-2}}(F(\vec{x})) F^* g_{\mathbb{R}^n}(\vec{\alpha}, \vec{\beta}).$$

Likewise,

$$F^*((w^\lambda)^{\frac{4}{n-2}} g_{\mathbb{R}^n})(\vec{\alpha}, \vec{\beta}) = (w^\lambda)^{\frac{4}{n-2}}(F(\vec{x})) g_{\mathbb{R}^n}(dF\vec{\alpha}, dF\vec{\beta})$$
$$= w^{\frac{4}{n-2}}(F(\vec{x})^\lambda) g_{\mathbb{R}^n}(dF\vec{\alpha}, dF\vec{\beta}) = w^{\frac{4}{n-2}}(F(\vec{x})^\lambda) F^* g_{\mathbb{R}^n}(\vec{\alpha}, \vec{\beta}).$$

Recall that for $\vec{x} = (x', x_n) \in \mathbb{R}^n$, $F^{-1}(\vec{x}) = (\frac{2x'}{1+|\vec{x}|^2}, \frac{2x_n}{1+|\vec{x}|^2}, \frac{|\vec{x}|^2-1}{|\vec{x}|^2+1})$. Thus the plane $\{\vec{x} : x_n \geq 0\}$ maps to a hemisphere under F^{-1}.

Thus we have the following results: z and z^λ are defined on $\mathbb{S}^n_+ \setminus F^{-1}(\tilde{\Gamma}) \times [0,T]$, where $\mathbb{S}^n_+$ is a hemisphere. z, z^λ satisfy the partial differential equation[17]

$$\frac{\partial u^{\frac{n+2}{n-2}}}{\partial t} = L_{g_{\mathbb{S}^n}} u + c(n) s u^{\frac{n+2}{n-2}}.$$

Moreover, from their definitions, it follows that $z^\lambda \leq z$ and $z = z^\lambda$ on $\partial\mathbb{S}^n_+$.

Our asymptotic expansions for $w(\cdot, t_0)$ and $w_{x_i}(\cdot, t_0)$, when viewing $\vec{y}(t_0)$ as the origin, imply that $\frac{\partial z(\cdot,t_0)}{\partial\nu}(\text{ north pole }) = \frac{\partial z^\lambda(\cdot,t_0)}{\partial\nu}(\text{ north pole }) = 0$, where ν denotes the inward unit normal of $\partial\mathbb{S}^n_+$. Thus, by the Hopf boundary point lemma we then get that $z \equiv z^\lambda$. Thus we get $w \equiv w^\lambda$, which is a contradiction. Thus we conclude that $\lambda > \max_{0\leq t\leq T} y_n(t)$, which implies that $\lambda \in I$. Thus I is closed, which implies $I = (\lambda_0, \infty)$. Therefore $y_n(T) \leq \lambda_0$ and thus $|\vec{y}(T)| \leq C$ for some C.
Q.E.D.

Therefore we get the following theorem

Theorem 7.2.8 *Assume that $(M, [g_0])$ is locally conformally flat and that $[g_0]$ is scalar positive. Choose a background metric $g_0 \in [g_0]$. If g is a solution to (1.2) with initial metric $g^0 \in [g_0]$ and u denotes the corresponding solution to (2.1), then*

$$\sup \frac{|\nabla_{g_0} u|}{u} \leq C, \tag{7.7}$$

[16]Via changing our coordinate system by a translation $x \mapsto x - y(t_0)$, if necessary.

[17]Since w does and F is a diffeomorphism

where $C > 0$ is a constant depending only on g^0, g_0 and the conformal properties of $(M, [g_0])$. For each t, integrating (7.7) along a shortest geodesic between a maximum point and a minimum point of $u(\cdot, t)$ yields

$$\inf u(\cdot, t) \geq \tilde{C} \sup u(\cdot, t) \tag{7.8}$$

where $\tilde{C} > 0$ is a constant.

Proof: By above results, we have that $|\vec{y}(T)| \leq C$, where

$$y_i(T) = \frac{\frac{\partial(\tilde{u}(\cdot,t)\circ G)(0)}{\partial x_i}}{(n-2)\tilde{u}(q_0, t)}$$

and $G(0) = q_0 \in \mathbb{S}^n$, q_0 is any point in $\Phi(V)$, and V is any neighborhood of some fixed point in $\tilde{M}$ so that $dist(\Phi(V), \partial\Omega) > 0$. Hence we get that

$$\frac{|\nabla_{g_{\mathbb{S}^n}} \tilde{u}|}{\tilde{u}} \leq C$$

holds on $\Phi(V')$ for some C, where $V' \subset\subset V$. This then gives us

$$\frac{|\nabla_{g_0} u|}{u} \leq C$$

on $\pi(V')$ for some C. Since M is compact, we can cover M by finitely many such V', and therefore obtain the last estimate on all of M by just using a larger C.

Thus we have that there exists a C so that

$$\frac{|\nabla_{g_0} u|}{u} \leq C.$$

Now consider a shortest geodesic curve r between two points $\vec{A}$, $\vec{B}$ such that $u(\vec{A}, t) = \inf u(\cdot, t)$, $u(\vec{B}, t) = \sup u(\cdot, t)$, $r(0) = \vec{A}$ and $r(1) = \vec{B}$. We get that

$$\log \frac{u(\vec{B}, t)}{u(\vec{A}, t)} = \int_0^1 \frac{\frac{d}{d\tau} u(r(\tau), t)}{u(r(\tau), t)} d\tau$$

$$= \int_0^1 \frac{< \nabla_{g_0} u(r(\tau), t), r'(\tau) >}{u(r(\tau), t)} d\tau \leq \int_0^1 \frac{|\nabla_{g_0} u|}{u} |r'| d\tau \leq C,$$

for some C. This then gives us that for some $\tilde{C}$ we have

$$u(\vec{B}, t) = \sup u(\cdot, t) \leq \tilde{C} u(\vec{A}, t) = \tilde{C} \inf u(\cdot, t).$$ [18]

Q.E.D.

Thus we get

[18] In all of the above arguments we assumed that $(M, [g_0])$ was not conformally covered by $\mathbb{S}^n$. However if it was, the same arguments still apply, we just have $\partial\Omega = \phi$ in this case.

Theorem 7.2.9 *Assume that $[g_0]$ is scalar positive, and $(M,[g_0])$ is locally conformally flat. Then for any given initial metric in $g^0 \in [g_0]$, the flow (1.2) has a unique smooth solution on the time interval $[0,\infty)$. Moreover, the solution metric g converges smoothly to a unique limit metric of constant scalar curvature as $t \to \infty$.*

Proof: let g be the unique solution to

$$\frac{\partial}{\partial t} g = (s-R)g,$$

$$g|_{t=0} = g^0$$

on $M \times [0,T^*)$, where we then write $g = u^{\frac{4}{n-2}} g_0$.

We know that $V(t) \equiv V(g^0)$, a constant, where $V(t) = \int_M dV_{g(\cdot,t)} = \int_M u^{\frac{2n}{n-2}} dV_{g_0}$. By the above Harnack inequality $\inf u(\cdot,t) \geq C \sup u(\cdot,t)$ we will get that u is bounded from above and bounded from below by a positive constant. This follows since

$$V(g^0) = \int_M u^{\frac{2n}{n-2}}(\cdot,t) dV_{g_0} \geq \inf u^{\frac{2n}{n-2}}(\cdot,t) V(g_0) \geq C^{\frac{2n}{n-2}} V(g_0) \sup u^{\frac{2n}{n-2}}(\cdot,t).$$

Therefore,

$$\sup u^{\frac{2n}{n-2}}(\cdot,t) \leq \frac{V(g^0)}{V(g_0)} C^{-\frac{2n}{n-2}}$$

which implies

$$\sup u(\cdot,t) \leq \frac{1}{C} \cdot \left(\frac{V(g^0)}{V(g_0)}\right)^{\frac{n-2}{2n}}.$$

Thus u is bounded from above. To show that u is bounded from below by a positive constant we note that

$$V(g^0) = \int_M u^{\frac{2n}{n-2}}(\cdot,t) dV_{g_0} \leq \sup u^{\frac{2n}{n-2}}(\cdot,t) V(g_0) \leq C^{-\frac{2n}{n-2}} \inf u^{\frac{2n}{n-2}}(\cdot,t) V(g_0)$$

which implies

$$\inf u^{\frac{2n}{n-2}}(\cdot,t) \geq \frac{V(g^0)}{V(g_0)} C^{\frac{2n}{n-2}}.$$

Thus

$$\inf u(\cdot,t) \geq C \cdot \left(\frac{V(g^0)}{V(g_0)}\right)^{\frac{n-2}{2n}}.$$

Therefore, u is bounded from below by a positive constant. Therefore, T^* must be ∞, otherwise we may extend our solution by local existence results to a larger time interval.

By Simon's results in [17], we get that u converges smoothly to a unique limit u_∞ as $t \to \infty$. We also know that

$$s'(t) = -\frac{n-2}{2V} \int (R-s)^2 dV$$

which implies that

$$\int_0^\infty \int (R-s)^2 dV dt < \infty.$$

From this we conclude that R_{g_∞} is constant, where $g_\infty = u_\infty^{\frac{4}{n-2}} g_0$.

Appendix A

The Variation of Certain Geometric Quantities

A.1 The Variation of the Volume Form

Let g be a Riemannian metric, and h a symmetric $(2,0)-$tensor, and $x^1, \cdots, x^n$ a local coordinate system.

We define $g(t) := g + th$, then we look at $dV_{g(t)} = \sqrt{\det(g(t)_{ij})}dx^1 \wedge \cdots \wedge dx^n$. Now,

$$\frac{d}{dt}dV_{g(t)}|_{t=0} = \frac{1}{2}\frac{1}{\sqrt{\det(g_{ij})}}\frac{d}{dt}\det(g_{ij} + th_{ij})|_{t=0}dx^1 \wedge \cdots \wedge dx^n.$$

Given a matrix A, we have

$$\frac{d}{dt}\det A = \det A \text{trace}[A^{-1}\frac{d}{dt}A].$$

Therefore,

$$\begin{aligned}\frac{d}{dt}dV_{g(t)}|_{t=0} &= \frac{1}{2}\frac{1}{\sqrt{\det(g_{ij})}}\det(g_{ij})\text{trace}(g^{ik}h_{kj})dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{2}\sqrt{\det(g_{ij})}g^{ik}h_{ki}dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{2}\sqrt{\det(g_{ij})}\text{tr}_g(h)dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{2}\text{tr}_g(h)dV_g.\end{aligned}$$

A.2 The Inverse of a Matrix

For $G = (g_{ij})_{1 \le i,j \le n}$, $H = (h_{ij})_{1 \le i,j \le n}$, we define $G(t) = G + tH$. For $G^{-1} = (g^{ij})$ we define $h^{ij} = \sum_{k,l} g^{ki}g^{lj}h_{kl}$. For t sufficiently small we have

$$(G + tH)^{-1} = (I + tG^{-1}H)^{-1}G^{-1}$$

$$= \sum_{n=0}^{\infty} (-tG^{-1}H)^n G^{-1}$$

$$= (I - tG^{-1}H + t^2(G^{-1}H)^2 + \cdots)G^{-1}$$

$$= (G^{-1} - tG^{-1}HG^{-1} + t^2(G^{-1}H)^2G^{-1} + \cdots)$$

$$G^{-1} - tG^{-1}HG^{-1} + O(t^2)\tilde{I},$$

where $\tilde{I}$ is the matrix whose entries are all 1.

Therefore,

$$\frac{d}{dt}G(t)^{-1}|_{t=0} = \frac{d}{dt}(G^{-1} - tG^{-1}HG^{-1} + t^2(G^{-1}H)^2G^{-1} + \cdots)|_{t=0}$$

$$= (-G^{-1}HG^{-1} + 2t(G^{-1}H)^2G^{-1} + \cdots)|_{t=0}$$

$$= -G^{-1}HG^{-1}.$$

Another way to see this is to use the fact that $G(t) \cdot G(t)^{-1} = I$. Thus,

$$\frac{d}{dt}(G(t) \cdot G(t)^{-1})|_{t=0} = 0$$

and thus

$$G \cdot \frac{d}{dt}G(t)^{-1}|_{t=0} + \frac{d}{dt}G(t)|_{t=0} \cdot G^{-1} = 0.$$

Thus, $G \cdot \frac{d}{dt}G(t)^{-1}|_{t=0} + HG^{-1} = 0$ which implies

$$\frac{d}{dt}G(t)^{-1}|_{t=0} = -G^{-1}HG^{-1}.$$

Now given a matrices $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$. Then let $(d_{ij}) = D = ABC$. Then $d_{ij} = \sum_{k,l} a_{il}b_{lk}c_{kj}$. Thus, using this for $A = C = G^{-1}$ and $B = H$ we get that

$$g(t)^{ij} = g^{ij} - t\sum_{kl} g^{il}h_{lk}g^{kj} + O(t^2) = g^{ij} - th^{ij} + O(t^2),$$

and thus

$$\frac{d}{dt}g(t)^{ij}|_{t=0} = -h^{ij}.$$

A.3 The Variation of the Christoffel Symbols

Recall that $\Gamma_{ij}^k = \frac{1}{2}\sum_l g^{kl}\{\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{il} - \frac{\partial}{\partial x_l}g_{ij}\}$. In the following discussion we shall assume that at a fixed point $p \in M$ we are working in a normal coordinate system. Thus at p we have $\Gamma_{ij}^k \equiv 0$ and $0 = \nabla_i g_{jk} = \frac{\partial}{\partial x_i}g_{jk}$.

Let $g_{ij}(t) = g_{ij} + th_{ij}$. Then quantities with respect to t are with respect to the metric $g_{ij}(t)$. Thus at $p \in M$ we have

$$\begin{aligned}
\Gamma(t)_{ij}^k - \Gamma_{ij}^k &= \frac{1}{2}\sum_l g(t)^{kl}\{\frac{\partial}{\partial x_i}g(t)_{jl} + \frac{\partial}{\partial x_j}g(t)_{il} - \frac{\partial}{\partial x_l}g(t)_{ij}\} \\
&= \frac{t}{2}\sum_l g(t)^{kl}\{\frac{\partial}{\partial x_i}h_{jl} + \frac{\partial}{\partial x_j}h_{il} - \frac{\partial}{\partial x_l}h_{ij}\} \\
&= \frac{t}{2}\sum_l g(t)^{kl}\{\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\}.
\end{aligned}$$

A.4 The Variation of the Ricci Tensor

Now in local coordinates, we have

$$R_{ij} = \sum_k\{\frac{\partial}{\partial x_k}\Gamma_{ij}^k - \frac{\partial}{\partial x_j}\Gamma_{ki}^k + \sum_l(\Gamma_{kl}^k\Gamma_{ji}^l - \Gamma_{jl}^k\Gamma_{ki}^l)\}.$$

$$R(t)_{ij} - R_{ij} = \sum_k\{\frac{\partial}{\partial x_k}(\Gamma(t)_{ij}^k - \Gamma_{ij}^k) - \frac{\partial}{\partial x_j}(\Gamma(t)_{ki}^k - \Gamma_{ki}^k)$$

$$+\sum_l(\Gamma(t)_{kl}^k\Gamma(t)_{ji}^l - \Gamma_{kl}^k\Gamma_{ji}^l) - \sum_l(\Gamma(t)_{jl}^k\Gamma(t)_{ki}^l - \Gamma_{jl}^k\Gamma_{ki}^l)\}.$$

Using the notation $C(t)_{ij}^k = \Gamma(t)_{ij}^k - \Gamma_{ij}^k$, we get

$$\begin{aligned}
R(t)_{ij} - R_{ij} = \sum_k\{&\partial_k C(t)_{ij}^k - \partial_j C(t)_{ki}^k + \sum_l(C(t)_{kl}^k\Gamma(t)_{ji}^l + \Gamma_{kl}^k C(t)_{ji}^l) \\
&- \sum_l(C(t)_{jl}^k\Gamma(t)_{ki}^l\Gamma_{jl}^k C(t)_{ki}^l)\}.
\end{aligned}$$

Now, assuming that we are using normal coordinates at a fixed point $p \in M$ (with respect to the metric g of course), we get

$$R(t)_{ij} - R_{ij} = \sum_k\{\nabla_k C(t)_{ij}^k - \nabla_j C(t)_{ki}^k + \sum_l C(t)_{kl}^k C(t)_{ji}^l - \sum_l C(t)_{jl}^k C(t)_{ki}^l\}.$$

Since this quantity is invariant with respect to changes in local coordinates, we get that this result must hold in general.

Thus,

$$\frac{d}{dt}R(t)_{ij}|_{t=0} = \frac{d}{dt}(R(t)_{ij} - R_{ij})|_{t=0}$$

$$= \sum_k \{\nabla_k(\frac{d}{dt}C(t)^k_{ij}|_{t=0}) - \nabla_j(\frac{d}{dt}C(t)^k_{ki}|_{t=0})\}$$

$$= \sum_k \{\nabla_k(\frac{d}{dt}\Gamma(t)^k_{ij}|_{t=0}) - \nabla_j(\frac{d}{dt}\Gamma(t)^k_{ki}|_{t=0})\}$$

$$= \frac{1}{2}\sum_{k,l}\{\nabla_k(g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})) - \nabla_j(g^{kl}\nabla_i h_{kl})\}$$

$$= \frac{1}{2}\sum_{k,l}\{g^{kl}\nabla_k(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}) - \nabla_j\nabla_i(g^{kl}h_{kl})\}$$

$$= \frac{1}{2}\sum_l \nabla^l(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}) - \frac{1}{2}\sum_l \nabla_j\nabla_i h^l_l.$$

A.5 The Variation of the Scalar Curvature

$$\frac{d}{dt}R_{g(t)}|_{t=0} = \frac{d}{dt}\{\sum_{ij} g(t)^{ij}R(t)_{ij}\}|_{t=0}$$

$$= \sum_{i,j}\{g^{ij}(\frac{d}{dt}R(t)_{ij}|_{t=0}) - h^{ij}R_{ij}\}$$

$$= \sum_{i,j,l} g^{ij}(\nabla^l(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}) - \nabla_j\nabla_i h^l_l) - \sum_{ij} h^{ij}R_{ij}$$

$$= \sum_{i,j}(\nabla^i\nabla^j h_{ij} - \nabla^j\nabla_j h^i_i - h^{ij}R_{ij}).$$

A.6 The Variation of $\int_M R_g dV_g$

$$\frac{d}{dt}\int_M R_{g(t)}dV_{g(t)}|_{t=0}{}^{1} = \int_M(\frac{d}{dt}R_{g(t)}|_{t=0})dV_g + \frac{1}{2}\int_M R_g \mathrm{tr}_g(h)dV_g$$

$$= \int_M \sum_{i,j}(\nabla^i\nabla^j h_{ij} - \nabla^j\nabla_j h^i_i - h_{ij}R^{ij})dV_g + \frac{1}{2}\int_M R_g \mathrm{tr}_g(h)dV_g$$

$$= \int_M \sum_{i,j}(-R^{ij} + \frac{1}{2}R_g g^{ij})h_{ij}dV_g$$

$$= - < h, \mathrm{Ric}(g) - \frac{1}{2}R_g \cdot g >_g .$$

[1]Here we assume that $\partial M = \phi$.

A.7 The Variation of $S(g) = V(g)^{\frac{2-n}{n}} \int_M R_g dV_g$

$$\frac{d}{dt} S(g+th)|_{t=0} = \frac{d}{d} \frac{\int_M R_{g(t)} dV_{g(t)}}{(\int_M dV_{g(t)})^{\frac{n-2}{n}}}|_{t=0}$$

$$= \frac{\frac{d}{dt}\int_M R_{g(t)} dV_{g(t)}|_{t=0}}{(\int_M dV_{g(t)})^{\frac{n-2}{n}}} + \frac{2-n}{2n}(\int_M dV)^{\frac{2-2n}{n}} \int_M R_g dV_g \int_M < h, g > dV_g$$

$$= \frac{< h, \frac{1}{2}R_g \cdot g - \mathrm{Ric}(g) - \frac{n-2}{2n} s_g \cdot g >_g}{V(g)^{\frac{n-2}{n}}}.$$

A.8 The Variation of $S(g)$ When Restricted to a Conformal Class

Let h, k be symmetric $(2,0)$-tensors. We now define the following inner product $< h, k >_{L^2(M,g)} = \int_M \sum_{i,j} h_{ij} k^{ij} dV_g$, where $k^{ij} = \sum_{l,m} g^{il} g^{jm} k_{lm}$. We define some functionals on $[g]$ by

$$V(g) = \int_M dV_g,$$

$$F(g) = \int_M R_g dV_g,$$

$$S(g) = \frac{\int_M R_g dV_g}{(\int_M dV_g)^{\frac{n-2}{n}}}.$$

We will now compute the Gateaux derivative of V. We take a variation of the metric

$$g + t\tilde{g} = (1 + tu^{\frac{4}{n-2}})g =: w^{\frac{4}{n-2}} g.$$

Thus,

$$V'(g)\tilde{g} = \frac{d}{dt} V(g + t\tilde{g})|_{t=0}$$

$$= \frac{d}{dt} \int_M w^{\frac{2n}{n-2}} dV_g|_{t=0}$$

$$= \frac{d}{dt} \int_M (1 + tu^{\frac{4}{n-2}})^{\frac{n}{2}} dV_g|_{t=0}$$

$$= \frac{n}{2} \int_M (1 + tu^{\frac{4}{n-2}})^{\frac{n-2}{2}}|_{t=0} u^{\frac{4}{n-2}} dV_g$$

$$= \frac{n}{2} \int_M u^{\frac{4}{n-2}} dV_g$$

$$= \frac{1}{2} \int_M \sum_{ij} g^{ij} u^{\frac{4}{n-2}} g_{ij} dV_g$$

$$=< \frac{1}{2} g, u^{\frac{4}{n-2}} g >_{L^2(M,g)} .$$

Next we compute the Gateaux derivative of F [2]

$$F'(g)\tilde{g} = \frac{d}{dt}F(g+t\tilde{g})|_{t=0}$$

$$= \frac{d}{dt}\int_M \frac{4(n-1)}{n-2} < \nabla_g w, \nabla_g w > + R_g w^2 dV_g|_{t=0}$$

$$= 2\int_M \frac{4(n-1)}{n-2} < \nabla_g w, \nabla_g \frac{dw}{dt} > + R_g w \frac{dw}{dt} dV_g|_{t=0}$$

$$= 2\int_M R_g \frac{dw}{dt}|_{t=0} dV_g$$

$$= 2\int_M R_g \cdot \frac{n-2}{4} \cdot (1+tu^{\frac{4}{n-2}})^{\frac{n-6}{4}}|_{t=0} \cdot u^{\frac{4}{n-2}} dV_g$$

$$= \frac{n-2}{2}\int_M R_g u^{\frac{4}{n-2}} dV_g$$

$$= \frac{n-2}{2n}\int_M \sum_{ij} R_g g^{ij} u^{\frac{4}{n-2}} g_{ij} dV_g$$

$$= < \frac{n-2}{2n} R_g g, u^{\frac{4}{n-2}} g >_{L^2(M,g)} .$$

Now we compute the Gateaux derivative of S

$$S'(g)\tilde{g} = \frac{d}{dt}S(g+t\tilde{g})|_{t=0}$$

$$= \frac{F'(g)\tilde{g}}{(V(g))^{\frac{n-2}{n}}} - \frac{n-2}{n}\frac{F(g)}{V(g)}\frac{V'(g)\tilde{g}}{(V(g))^{\frac{n-2}{n}}}$$

$$= \frac{n-2}{2}(V(g))^{\frac{2-n}{n}}[\int_M R_g u^{\frac{4}{n-2}} dV_g - \frac{F(g)}{V(g)}\int_M u^{\frac{4}{n-2}} dV_g]$$

$$= \frac{n-2}{2}(V(g))^{\frac{2-n}{n}}[\int_M R_g u^{\frac{4}{n-2}} dV_g - s_g\int_M u^{\frac{4}{n-2}} dV_g]$$

$$= \frac{n-2}{2}(V(g))^{\frac{2-n}{n}}\int_M (R_g - s_g) u^{\frac{4}{n-2}} dV_g$$

$$= \frac{n-2}{2n}(V(g))^{\frac{2-n}{n}}\int_M \sum_{ij} g_{ij}(R_g - s_g) u^{\frac{4}{n-2}} g^{ij} dV_g$$

$$= < \frac{n-2}{2n}(V(g))^{\frac{2-n}{n}}(R_g - s_g)g, u^{\frac{4}{n-2}} g >_{L^2(M,g)} .$$

Thus, restricting ourselves to $[g]$, and using $< \ , \ >_{L^2(M,g)}$ as our inner product, we have

$$\text{grad}V(g) = \frac{1}{2}g,$$

[2] Recall that $u^{\frac{n+2}{n-2}} R_{\tilde{g}} = -\frac{4(n-1)}{n-2}\triangle_g u + R_g u$.

$$\mathrm{grad}F(g) = \frac{n-2}{2n} R_g g,$$
$$\mathrm{grad}S(g) = \frac{n-2}{2n} (V(g))^{\frac{2-n}{n}} (R_g - s_g) g.$$

A.8.1 The Setup of [19]

Now we consider the functional $F(g)$. In [19], the flow $g_t = -\mathrm{grad}S(g)$ is studied. We want to now show that this can be seen as the gradient flow of $F(g)$ when projected onto the unit ball. Let's fix g with $V(g) = 1$. Then $< g, g >_{L^2(M,g)} = n$ and for $\Sigma := \{\tilde{g} : V(\tilde{g}) = 1, \tilde{g} \in [g]\}$, we have $g \perp T_g\Sigma$[3].

Now, the projection of $\mathrm{grad}F(g)$ onto g is

$$\begin{aligned} \frac{< \mathrm{grad}F(g), g >_{L^2(M,g)}}{< g, g >_{L^2(M,g)}} g &= \frac{1}{n} \int_M \sum_{ij} \frac{n-2}{2n} R_g g_{ij} g^{ij} dV_g \cdot g \\ &= \frac{n-2}{2n} \int_M R_g dV_g \cdot g \\ &= \frac{n-2}{2n} s_g \cdot g \end{aligned}$$

Thus

$$\mathrm{grad}F(g) - \frac{n-2}{2n} s_g g = \frac{n-2}{2n} (R_g - s_g) g.$$

Moreover $\mathrm{grad}F(g) - \frac{n-2}{2n} s_g g \perp \mathrm{grad}V(g)$.

[3] Here we actually use that $\mathrm{grad}V(g) \perp \Sigma$.

Appendix B

Some Results from [15]

M^n is ***locally conformally flat*** if there exists a covering $\{U_\alpha, \varphi_\alpha\}$ of M^n such that $\varphi_\alpha : U_\alpha \to \mathbb{S}^n$ and when $U_\alpha \cap U_\beta \neq \phi$, the change of coordinates map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a conformal diffeomorphism from $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$.

If M is locally conformally flat and g is a Riemannian metric on M, we say g is compatible with the conformally flat structure if for all α, $\varphi_\alpha : (U_\alpha, g) \to \mathbb{S}^n$ is a conformal mapping which is true if and only if $g \equiv \lambda(x) \sum_{i=1}^n (dx^i)^2$ for $\lambda > 0$ on U_α, where $\varphi_\alpha = (x^1, \cdots x^n)$.

A partition of unity argument shows that a paracompact, locally conformally flat manifold admits a compatible metric g (compatible with the conformally flat structure).

We note also the following important theorem and some of its corollaries.

Liouville Theorem B.0.1 *Every conformal map of an open subset U of $\mathbb{R}^n$ onto an open subset V of $\mathbb{R}^n$ is the restriction of a composition of similarities and inversions, in fact at most one of each. In addition, these conformal maps take hyperplanes and spheres to hyperplanes and spheres.*

Corollary B.0.2 *The conformal transformations of $\mathbb{S}^n$ are determined locally and given by Möbius transformations.*

Corollary B.0.3 *For $\{U_\alpha, \varphi_\alpha\}$ as above, $\varphi_\beta \circ \varphi_\alpha^{-1}$ on any connected component of $\varphi_\alpha(U_\alpha \cup U_\beta)$ is the restriction of a Möbius Transformation of $\mathbb{S}^n$.*

Suppose M is a smooth manifold and $\Phi : M \to \mathbb{S}^n$ is an immersion [1]. The immersion then induces a locally conformally flat structure on M, since at each point $p \in M$ there exists a neighborhood U_p of p such that $\Phi : U_p \to \mathbb{S}^n$ is a diffeomorphism onto its image. The change of coordinates transformation is then the identity, and hence M has a unique locally conformally flat structure with respect to which $\Phi : M \to \mathbb{S}^n$ is a conformal map.

If g_0 is the standard Riemannian metric on $\mathbb{S}^n$, $\Phi^* g_0$ is a compatible (incomplete) Riemannian metric on M. For a manifold M^n which is simply connected and $n \geq 3$, every locally conformally flat structure on M is induced by an immersion $\Phi : M^n \to \mathbb{S}^n$ called the ***developing map***.

[1] $d\Phi$ is a linear isomorphism at each point

To see this, observe that if $p_0 \in M$ and (U_0, φ_0) is a chart with $p_0 \in U_0$, then we can define $\Phi \equiv \varphi_0$ in a small neighborhood of p_0. We can then analytically continue Φ along any curve passing through p_0 in the following way: If Φ is defined on an open arc γ, and if $p \in M$ is an endpoint of γ, then we can choose a coordinate chart (U, φ) containing p and by Liouville's theorem we have a Möbius transformation $\psi : \mathbb{S}^n \to \mathbb{S}^n$ which agrees with $\Phi \circ \varphi^{-1}$ in a neighborhood of $\varphi(p)$. We can then extend Φ to a neighborhood of p by defining $\Phi \equiv \psi \circ \varphi$. Since M is simply connected (by assumption), we get a well-defined locally conformal map $\Phi : M \to \mathbb{S}^n$.

For a general locally conformally flat manifold M, we have the immersion $\Phi : \tilde{M} \to \mathbb{S}^n$, where $\tilde{M}$ is the ***universal covering manifold*** of M. [2]

If $\Phi : M \to S^n$ an immersion, then Φ induces a unique locally conformally flat structure on M. If $g_{\mathbb{S}^n}$ is the standard metric on $\mathbb{S}^n$, then $\Phi^* g_{\mathbb{S}^n}$ is a compatible metric on M. For M^n, $n \geq 3$ which is simply connected, every locally conformally flat structure on M is induced by an immersion $\Phi : M^n \to \mathbb{S}^n$ called the ***developing map***. For a general M, we have the immersion $\Phi : \tilde{M} \to \mathbb{S}^n$, here $\tilde{M}$ is the universal cover of M. If π is the covering map of $\tilde{M}$ onto M, then if g_0 represents a metric on M, we can pull back g_0 to get a metric $\tilde{g_0} := \pi^* g_0$ on $\tilde{M}$. Recall that the pull back metric $\tilde{g_0} = \pi^* g_0$ is defined by: for $\tilde{x} \in \tilde{M}$, $\pi x \in M$, take $\alpha, \beta \in T_{\tilde{x}} \tilde{M}$, we define $\tilde{g_0}(\alpha, \beta) = g_0(d\pi\alpha, d\pi\beta)$. Note that if $R_{g_0} \geq 0$, then $R_{\tilde{g_0}} \geq 0$ since locally they're the same.

The following is a theorem of [15] which gives a sufficient condition for the developing map to be injective.

Theorem from [15] B.0.4 *Let (M^n, g) $(n \geq 3)$ be a complete Riemannian manifold. Suppose that the scalar curvature R_g is bounded and non-negative. Let $\Phi : M \to \mathbb{S}^n$ a conformal map. Then Φ is injective and $\partial\Phi$ has zero Newtonian capacity. If $g_{\mathbb{S}^n}$ denotes the usual metric on $\mathbb{S}^n$, and $\Phi : \tilde{M} \to \mathbb{S}^n$ is the immersion mentioned above. Then Φ is a conformal diffeomorphism onto $\Omega \subset \mathbb{S}^n$, an open dense set. Thus $\Phi^* g_{S^n} = f\tilde{g_0}$, for some $f > 0$.*

[2] Let $p : E \to B$ be a continuous surjection. An open set $U \subset B$ is said to be ***evenly covered*** by p if $p^{-1}(U)$ is a disjoint union $\cup V_\alpha$, $V_\alpha \subset E$ such that for any α, $p|_{V_\alpha}$ is a homeomorphism onto U. $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into slices. If every $b \in B$ has an open neighborhood U which is evenly covered by p, then p is a covering map and E is said to be a covering space of B. To eliminate trivial coverings of the "pancake-stack" variety, one often requires E to be connected. If $p : E \to B$ is a covering map, then $p : E \to B$ is a local homeomorphism, but the converse is not necessarially true. If E is simply connected and $p : E \to B$ is a covering map, then E is said to be a ***universal covering space*** of B.

Bibliography

[1] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer, Berlin, 1998.

[2] A. Besse, *Einstein Manifolds*, Springer, Berlin, 1987.

[3] S. Brendle, *The Generalization of the Yamabe Flow for Manifolds with Boundary*, Asian J. Math. 6 (2002) 625-644.

[4] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and Related Properties via the Maximum Principle*, Comm. Math. Phys. 68 (1979) 209-243.

[5] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 2001.

[6] R.S. Hamilton, *The Ricci Flow on Surfaces*, Mathematics and General Relativity, Contemporary Math, Vol. 71, Amer. Math. Soc., Providence, RI, 1988, 237-262.

[7] R.S. Hamilton, *Three-Manifolds with Positive Ricci Curvature*, J Diff. Geom. 17 (1982) 255-306.

[8] R.S. Hamilton, *Four-Manifolds with Positive Curvature Operator*, J. Diff. Geom. 24 (1986) 153-179.

[9] N.V. Krylov, *Nonlinear Elliptic and Parabolic Equations of the Second Order*, D. Reidel Publishing Company, Boston, 1987.

[10] N.V. Krylov, *Lectures on Elliptic and Parabolic Equations in Hölder Spaces*, AMS GSM Vol. 12, AMS, 1996.

[11] N.V. Krylov and M.V. Safanov, *A Certain Property of Solutions of Parabolic Equations with Measurable Coefficients*, Math USSR-Izv. 16 (1981) 151-164.

[12] J.M. Lee and T.H. Parker, *The Yamabe Problem*, Bulletin of AMS 17 (1987) 37 - 91.

[13] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, New Jersey, 1996.

[14] J. Munkres, *Topology, a First Course*, Prentice Hall, Englewood Cliffs, NJ, 1975.

[15] R. Schoen and S.T. Yau, *Conformally Flat Manifolds, Kleinian Groups and Scalar Curvature*, Invent. math. 92 (1988) 47 - 71.

[16] R. Schoen and S.T. Yau, *Lectures on Differential Geometry*, International Press, Inc., Cambridge, MA, 1994.

[17] L. Simon, *Asymptotics for a Class of Nonlinear Evolution Equations, with Applications to Geometric Problems*, Ann. of Math. (2) 118 (1983) 525-571.

[18] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, Inc., Houston, Texas, 1999.

[19] R. Ye, *Global Existence and Convergence of Yamabe Flow*, J. Diff. Geom. 39 (1994) 35-50.