

Prescribing Gauss-Kronecker Curvature on Group Invariant Convex Hypersurfaces

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Given a positive G -invariant function, defined in $\mathbb{R}^{n+1} \setminus \{0\}$ where $G \subseteq O(n+1)$, we would like to find a hypersurface M whose Gauss-Kronecker Curvature is Given by $F|_M$.

Background Material:

- Let $M \subset \mathbb{R}^{n+1}$ be a smooth, orientable, regular hypersurface, i.e. M is locally a solution to an equation of the type

$$G(x_1, \dots, x_{n+1}) = 0,$$

with

$$\nabla G(x_1, \dots, x_{n+1}) \neq 0.$$

- M is **orientable** if \exists differentiable $N : M \rightarrow \mathbb{S}^n = \{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$. N is called the **Gauss Map**.
- $dN_p : T_p M \rightarrow T_{N(p)} \mathbb{S}^n \cong T_p M$
- The eigenvalues $\kappa_1(p), \dots, \kappa_n(p)$ of $-dN_p$ are called the **Principal Curvatures** of M at p .

- $\Pi_p(\cdot, \cdot) = \langle -dN_p(\cdot), \cdot \rangle: T_pM \times T_pM \rightarrow \mathbb{R}$ is called the **Second Fundamental Form**.
- If $\{e_1, \dots, e_n\}$ is an orthonormal basis for T_pM , then the eigenvalues of $\Pi_p(e_i, e_j)$ are the principal curvatures.
- If M is expressed locally as a graph $(\vec{x}, u(\vec{x}))$ around $p = 0$, with $u(0) = 0$, $\nabla u(0) = 0$, then the eigenvalues of $D^2u(0)$ are the principal curvatures of M at p .
- **Gauss-Kronecker Curvature** of M at p :

$$K(p) = \kappa_1(p) \cdots \kappa_n(p)$$

Prescribing Gauss-Kronecker Curvature:

Given $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^+$, what hypersurfaces M satisfy

$$K(\vec{X}) = F(\vec{X})$$

$\forall \vec{X} \in M$?

- By a theorem of **Hadamard**, any hypersurface M with $K(\vec{X}) = F(\vec{X})$, $\forall \vec{X} \in M$, must be convex.
- In 1979-80, S.T. Yau raised the question of sufficient conditions to be placed on F for the existence of such a hypersurface.

- Case $F \equiv 1$: $K_{\mathbb{S}^n} \equiv 1$. It can be shown that \mathbb{S}^n is the only solution. For uniqueness results of this type, see:
 - Aepli (Proc. Amer. Math. Soc. '60)
 - A.D. Alexandrov (AMS Transl. '62)
 - Ros and Korevaar (JDG '88)
 - Li (Cont. Math. '97)

- Similar problems with other Weingarten curvatures were also studied extensively. In the setting of Euclidean space as the ambient space, see for instance:
 - Caffarelli, Nirenberg and Spruck (Current Topics in PDE's '86): Prescribing more general symmetric functions of the principal curvatures.
 - Bakelman and Kantor (Geom. and Top., Leningrad '74), Treibergs and Wei (JDG '83): Prescribing mean curvature.

- To find M such that $K(\vec{X}) = F(\vec{X}) \forall \vec{X} \in M$, we use tools of **fully non-linear second order elliptic partial differential equations**.

- We look for M star-shaped of the form

$$M = \{\rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^n\}.$$

- Then $u = \rho^{-1}$ must satisfy:

$$\frac{u^{n+2}}{(u^2 + |\nabla u|^2)^{\frac{n+2}{2}}} \frac{\det(\nabla_{ij}u + ue_{ij})}{\det(e_{ij})} = F\left(\frac{1}{u(\vec{x})}\vec{x}\right)$$

– $e_{ij} = \langle e_i, e_j \rangle$ is the usual metric for \mathbb{S}^n

– e_1, \dots, e_n is a local frame

– ∇ is covariant differentiation on \mathbb{S}^n

- This equation for u is of **Monge-Ampère** type.

The Method of Continuity Approach:

THEOREM (Oliker Comm. PDE '84) 1 *Let F be a smooth positive function defined on the set $\{\vec{X} \in \mathbb{R}^{n+1} : r_1 \leq |\vec{X}| \leq r_2\}$, $r_1 < 1 < r_2$*

$$F(\vec{X})|\vec{X}|^n \geq 1 \quad \forall |\vec{X}| = r_1, \quad (1)$$

$$F(\vec{X})|\vec{X}|^n \leq 1 \quad \forall |\vec{X}| = r_2, \quad (2)$$

and

$$\frac{\partial}{\partial \rho}(\rho^n F(\rho \vec{x})) \leq 0 \quad (3)$$

for each fixed $\vec{x} \in \mathbb{S}^n$. Then, there exists a smooth hypersurface M with $K(\vec{X}) = F(\vec{X})$ $\forall \vec{X} \in M$. Moreover, any two solutions are endpoints of a one parameter family of homothetic dilations, all of which are solutions to $F(\vec{X}) = K(\vec{X})$.

Oliker used the so-called **method of continuity**.

- In solving an equation of the type

$$H(\vec{x}, u, \nabla u, \nabla^2 u) = 0,$$

we need that the linearized operator

$$Lw = H_{\nabla^2 u} \nabla^2 w + H_{\nabla u} \nabla w + H_u w$$

at a solution u , is invertible.

- For this, we only need to check that the null space of L is 0.

Oliker needed $\frac{\partial}{\partial \rho}(\rho^n F(\rho \vec{x})) \leq 0$ for:

- To conclude that the null space is zero.
- To obtain the uniqueness up to homothety.

Delanoë (Ann. Sci. Ecole Norm Sup. '85) was able to drop this monotone condition at the expense of uniqueness using the **fixed point theorem of Leray-Schauder**.

A Parabolic Flow Approach:

Tso (JDG '91) raised the question: Prescribing K when F bounded between two positive constants.

- Note that this case includes the case where F is constant, and thus is a generalization of the results established by Aeppli, A.D. Alexandrov, et al.
- Oliker's result does not cover this case. In fact Oliker's result doesn't even address the case of F being a constant.

Tso (JDG '91) proved the following results by a curvature flow approach:

- For $F \in L^1(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1})$, $F > 0$ there exists a convex hypersurface M whose Gauss-Kronecker \vec{X} curvature satisfies $K(\vec{X}) = F(\vec{X})$ for $\vec{X} \in M$ if and only if there is a smooth uniformly convex hypersurface N so that

$$J(N) := \int_N \sigma_{n-1} - \int_{\tilde{N}} F \leq 0$$

where σ_{n-1} is the normalized $(n-1)^{st}$ elementary symmetric function of the principal curvatures, and \tilde{N} is the convex body whose boundary is N .

- Tso proved this by considering the logarithmic gradient flow

$$\frac{dX}{dt} = -(\log K - \log F)\nu$$

where ν is the outer unit normal, to the hypersurface $X(\cdot, t)$.

– Here we have

$$\frac{d}{dt}J(X(\cdot, t)) = - \int_{X(\cdot, t)} (K - F) \log \frac{K}{F} \leq 0.$$

– Tso's techniques also reproved Olikier's work in Comm. PDE '84, without the monotonicity assumption (3), and by using the logarithmic Gauss Curvature flow.

- For $F \in C^\infty(\mathbb{R}^{n+1})$ which satisfies $\Omega = \{\vec{x} : F(\vec{x}) > 0\}$ is bounded and $F^{\frac{1}{n}}$ concave in Ω and the ratio between the minimum and maximum eigenvalues of $\text{Hess}F^{\frac{1}{n}}$ is uniformly from above, there there are two solutions to $K = F|_M$ for some M if there exists a convex hypersurface $N \subseteq \Omega$ satisfying $J(N) \leq 0$
 - Tso proved this result by considering the geometric flow

$$\frac{dX}{dt} = -(K^{\frac{1}{n}} - F^{\frac{1}{n}})\nu.$$

- Tso also considered the problem of prescribing Gauss-Kronecker curvature in the case where the given function F is group invariant with respect to a subgroup of $O(n+1)$.

– In this case, Tso assumed that G satisfies that

$$\{\vec{x} : \vec{x}g \text{ spans } \mathbb{R}^{n+1} \text{ for some } x\} \neq \phi.$$

– In the case where F is bounded between two constants (as well as a broader class of functions), Tso showed there is a hypersurface M so that $F|_M = K$.

Degree Theory Approach:

An alternate approach to the *method of continuity approach* when examining the Monge-Ampère elliptic PDE for this problem is via a **degree theory** argument.

- The advantage is you do not need to know of the invertibility of the linearized operator.
- Require that F is invariant under a fixed-point free $G \subset O(n + 1)$.

Li (Cont. Math. '97) showed: F bounded and G -invariant $\Rightarrow \exists M$ with Gauss-Kronecker curvature F .

Moreover, Li showed: F strictly monotone in one direction \Rightarrow There doesn't exist M with Gauss-Kronecker curvature $F|_M$.

- For example:

$$F(\vec{X}) = \tan^{-1}(X_{n+1}) + 10\pi$$

does not admit a solution. This was proven by a method of moving planes type of argument.

The following result appears in the *Journal of Differential and Integral Equations*, Oct. '06.

THEOREM (Mikula '06) 1 *Let $F \in C^{k,\alpha}(\mathbb{R}^3 \setminus \{0\})$, $k \geq 2$, $\alpha \in (0, 1)$, G -invariant, where $G \subset O(3)$ is a fixed point free. Suppose F also satisfies*

$$\limsup_{|\vec{X}| \rightarrow 0} F(\vec{X})|\vec{X}|^2 < 1,$$

$$\liminf_{|\vec{X}| \rightarrow \infty} F(\vec{X})|\vec{X}|^2 > 1.$$

Then there exists G -invariant $M := \{\rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$ with Gauss-Kronecker curvature $K(\vec{X}) = F(\vec{X}) \forall \vec{X} \in M$.

The Growth conditions on F are optimal:

- For $\varepsilon > 0$, the functions $F(\vec{X}) = \frac{(1 \pm \varepsilon)}{|\vec{X}|^2}$ do not admit M with $K(\vec{X}) = F(\vec{X}) \forall \vec{X} \in M$.
 - Suppose $M \subset \mathbb{R}^3$ has Gauss-Kronecker curvature

$$K(\vec{x}) = \frac{1 + \varepsilon}{|\vec{x}|^2} \quad \forall \vec{x} \in M.$$

- Let $\vec{z} \in M$ be such that $|\vec{z}| = \inf_{\vec{y} \in M} |\vec{y}|$. Let B be the ball of radius $|\vec{z}|$ and center $\vec{0}$.
- Then the Gauss-Kronecker curvature of M at \vec{z} satisfies:

$$K(\vec{z}) = \frac{1 + \varepsilon}{|\vec{z}|^2} \leq \frac{1}{|\vec{z}|^2}$$

$$\Rightarrow 1 + \varepsilon \leq 1.$$

Contradiction!

Open Problem: 1 Let $F \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$, G -invariant, $G \subset O(n+1)$ fixed-point free. Suppose F also satisfies

$$\limsup_{|\vec{X}| \rightarrow 0} F(\vec{X})|\vec{X}|^n < 1,$$

$$\liminf_{|\vec{X}| \rightarrow \infty} F(\vec{X})|\vec{X}|^n > 1.$$

Show that there exists a G -invariant hypersurface $M := \{\rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^n\}$ with Gauss-Kronecker curvature $K(\vec{X}) = F(\vec{X}) \forall \vec{X} \in M$.

Partial Result in \mathbb{R}^{n+1} (Mikula):

If we require some conditions on the fixed point free subgroup $G \subseteq O(n+1)$ and a growth condition at ∞ for F , we get the result:

- $\exists g \in G$ s.t. $g(0, \dots, 0, 1) = (0, \dots, 0, -1)$ and $g\{\vec{x} : x_{n+1} = 0\} \subset \{\vec{x} : x_{n+1} = 0\}$.
- $\exists e(G) \in (0, 1)$ s.t. $\forall \vec{x}, \vec{y} \in \mathbb{S}^n \cap \{\vec{z} : z_{n+1} = 0\}$ $\exists g \in G$ fixing $(0, \dots, 0, 1)$ s.t. $\frac{\langle \vec{y}, g\vec{x} \rangle}{\|\vec{y}\| \|\vec{x}\|} \geq e(G)$.
- $\limsup_{|\vec{x}| \rightarrow \infty} F(\vec{x}) < \infty$.

Procedure of Proof via the Degree Theory Argument:

For u a solution to

$$\frac{u^{n+2}}{(u^2 + |\nabla u|^2)^{\frac{n+2}{2}}} \frac{\det(\nabla_{ij}u + ue_{ij})}{\det(e_{ij})} = F\left(\frac{1}{u(\vec{x})}\vec{x}\right),$$

we need only establish a priori estimates.

- C^0 a priori estimates are the heart of the matter.
- C^0 estimates $\Rightarrow C^1$ estimates: Convexity
- C^0, C^1 estimates $\Rightarrow C^2$ estimates: Bernstein or Pogorelov type argument
- C^2 estimates $\Rightarrow C^{2,\alpha}$ estimates, $\alpha \in (0, 1)$: Evans (Comm. Pure Appl. Math '82) and Krylov (Izvestia Math '83)
- $C^{2,\alpha} \Rightarrow C^{4,\alpha}$: Standard elliptic theory (Schauder)

For C^0 a priori estimates:

- Use convexity of M .
- Group invariance of M .
- Constructed comparison surfaces and use of the comparison principle / maximum principle.

Degree theory for second order fully nonlinear elliptic operators on Riemannian manifolds was given in Li (Comm. in PDE '89).

- If degree $\neq 0$, then we obtain a solution.
- The degree is homotopy invariant.
- Let C be the constant so that $\frac{1}{C} < u < C$ and $\|u\|_{C^{4,\alpha}(\mathbb{S}^n)} < C$, and set

$$\mathcal{O} := \{u \in C_G^{4,\alpha}(\mathbb{S}^n) : \frac{1}{C} < u < C, \|u\|_{C^{4,\alpha}(\mathbb{S}^n)} < C\}.$$

- We want to solve:

$$h(\vec{x}, u, \nabla u, \nabla^2 u) = F\left(\frac{1}{u(\vec{x})}\vec{x}\right)$$

- Define:

$$\mathcal{F}(u, t) = h(\vec{x}, u, \nabla u, \nabla^2 u) - tF\left(\frac{1}{u(\vec{x})}\vec{x}\right) + t - 1$$

- We have

$$\deg(\mathcal{F}(\cdot, 1), \mathcal{O}, 0) = \deg(\mathcal{F}(\cdot, 0), \mathcal{O}, 0).$$

- $u \equiv 1$ is the unique positive solution of $\mathcal{F}(u, 0) = 0$ in $C_G^{4,\alpha}(\mathbb{S}^n)$.

- $\mathcal{F}_u(1, 0): C_G^{4,\alpha}(\mathbb{S}^n) \rightarrow C_G^{2,\alpha}(\mathbb{S}^n)$ is invertible.
 - Use: Kernel of $\Delta + n$ is span of spherical harmonics of degree 1. 0 is only G -invariant element of the kernel.

- Thus

$$\deg(\mathcal{F}(\cdot, 0), \mathcal{O}, 0) = (-1)^i,$$

where i is the number of negative eigenvalues of $\Delta + n$ in $C_G^{4,\alpha}(\mathbb{S}^n)$.

- $\deg(\mathcal{F}(\cdot, 1), \mathcal{O}, 0) \neq 0 \Rightarrow \mathcal{F}(u, 1) = 0$ has at least one solution in \mathcal{O} .

Some Results in the Setting of General Riemannian Manifolds:

C. Gerhardt (JDG '96 and Math. Z. '97) considered the problem of prescribed functions \mathcal{F} of the principal curvatures in the setting of general Riemannian manifold N .

- More specifically, for $\Omega \subset N$ open connected and $f \in C^{2,\alpha}(\bar{\Omega})$, Gerhardt showed existence of $M \subset \bar{\Omega}$ s.t. $\mathcal{F} = f|_M$.
- Assumed Barrier conditions on f .
- Assumed \mathcal{F} was of class (K), a large class of functions which includes $K^{\frac{1}{n}}$.
- Assume Ω can be covered by a normal Gaussian coordinate system, e.g. N with non-pos. sectional curvature; N space form.
- In JDG 43 Gerhardt used a **parabolic evolution method** and in Math. Z. 224 he used a method of **successive approximations** to solve the problem $\mathcal{F} = f|_M$ for some hypersurface M .

Some Results in the Setting of Space Forms:

Oliker (Cont. Math. '89) considered the problem in the setting of Space Forms, which are complete Riemannian manifolds of constant sectional curvature.

- Assumed barrier conditions and monotonicity conditions on f similar to that his result in Comm. PDE (1984).
- Was able to obtain results similar to that obtained in Euclidean space in the setting of Hyperbolic space forms.
- Used the method of continuity to show the existence of a unique closed convex hypersurface which is a radial graph over \mathbb{S}^n .

Y.Y. Li and V. Oliker (J. PDE '02) In the setting of elliptic space forms, considered the analogous problem of **prescribing normalized k -th mean curvature** using a **degree theory argument**.

- Assumed barrier conditions and monotonicity conditions on the prescribing function f , similar to Oliker's condition in Comm. PDE ('84).
- It is an open problem to prove the result in the case where we have growth conditions similar to those in (Mikula JDIE '06) in Elliptic space forms. In such a case it is only necessary to establish C^0 a priori estimates.

Y.Y. Li and Q. Jin (Disc. and Cont. Dyn. Sys. '06) extended the above result in \mathbb{H}^{n+1} using a **degree theory argument** for the analogous problem of **prescribing k – th mean curvature**.

- Assumed barrier conditions and monotonicity conditions on the prescribing function f , similar to Oliker's condition in Comm. PDE ('84).
- It is an open problem to prove the result in the case where we have growth conditions similar to those in (Mikula JDIE '06) in Hyperbolic space forms. In such a case it is only necessary to establish C^0 a priori estimates.

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