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PRESCRIBING GAUSS-KRONECKER CURVATURE ON GROUP INVARIANT CONVEX HYPERSURFACES

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Abstract. We consider the problem of prescribing Gauss-Kronecker curvature in Euclidean space. In particular, by a degree theory argument, we prove the existence of a closed convex hypersurface in $\mathbb{R}^3$ which has its Gauss-Kronecker curvature equal to F, a prescribed positive function, which is invariant under a fixed-point free subgroup G of the orthogonal group $O(3)$, requiring that F satisfy natural growth assumptions near the origin and at infinity.

1. Introduction

1.1. **The Weingarten Curvature Problem.** We consider the problem of the existence of a hypersurface in $\mathbb{R}^{n+1}$ with prescribed Weingarten curvature on radial directions. Recall that for a compact hypersurface $M \subset \mathbb{R}^{n+1}$, the k^{th} Weingarten curvature at $\vec{x} \in M$ is defined as

$$W_k(\vec{x}) = \sigma_k(\kappa_1(\vec{x}), \cdots, \kappa_n(\vec{x})), \tag{1.1}$$

where $\kappa = (\kappa_1, \cdots, \kappa_n)$ are the principal curvatures of M and

$$\sigma_k(\lambda_1, \cdots, \lambda_k) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \le k \le n, \tag{1.2}$$

is the k^{th} elementary symmetric function. Given a function $F : \mathbb{R}^{n+1} \to \mathbb{R}^+$, we would like to examine if there is a hypersurface $M \subset \mathbb{R}^{n+1}$, which is star-shaped about the origin, satisfying

$$W_k(\vec{x}) = F(\vec{x}) \quad \forall \vec{x} \in M. \tag{1.3}$$

When $k = n$, $W_n(\vec{x}) = K(\vec{x})$ is the so-called Gauss-Kronecker curvature of M at $\vec{x}$.

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1.2. **Some History of the Problem (1.3).** In 1979-80, S.T. Yau raised the question of sufficient conditions to be placed on $F : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ to insure the existence of a solution to (1.3) (see [25]). Since the early 1980's, problems of this type have been studied extensively.

The case of F being constant was studied by various mathematicians. In the case of $k = 1$, in [2] A.D. Alexandrov showed that an embedded solution to (1.3) must necessarily be a sphere. In the immersed case, there are known counterexamples. See for instance [10] and [24]. When $k = 2$, Ros [20] and Korevaar [12] showed that the sphere is the only embedded hypersurface with constant scalar curvature. In the case $k = n$, we may conclude via the Hadamard theorem that M is strictly convex. But if we know that W_k is constant for some k then we may conclude that M is a sphere, see for instance [11], [16] and [17].

When F is positive and defined on $\{\vec{x} \in \mathbb{R}^{n+1} : r_1 \leq |\vec{x}| \leq r_2\}$ and F satisfies $F(\vec{x})|\vec{x}|^k \geq C_k^n$ for all $|\vec{x}| = r_1$, $F(\vec{x})|\vec{x}|^k \leq C_n^k$ for all $|\vec{x}| = r_2$ and $\frac{\partial}{\partial \rho}(\rho^k F(\rho\vec{x})) \leq 0$ for all $\vec{x} \in \mathbb{S}^n$, then in the case $k = 1$, Treibergs and Wei [22] and Bakelman and Kantor [3] showed that there exists a solution to (1.3). If $k = n$, Oliker obtained a solution to (1.3), and in the general case Caffarelli, Nirenberg and Spruck [4] obtained a solution. The solutions obtained are unique up to homothety. In [5], Delanoë dropped the assumption $\frac{\partial}{\partial \rho}(\rho^n F(\rho\vec{x})) \leq 0$ and obtain a solution to (1.3) when $k = n$. However, uniqueness of solutions no longer holds in this case.

Oliker [18] and Caffarelli, Nirenberg and Spruck [4] used the so-called method of continuity to obtain their results. In order to apply the inverse function theorem, they needed to show that the linearized operator for (1.3) has trivial kernel. In order to do this, they required the condition $\frac{\partial}{\partial \rho}(\rho^k F(\rho\vec{x})) \leq 0$, and made use of the maximum principle. Moreover, this monotonicity condition also gives the uniqueness up to homothety via the maximum principle. In [23], Tso considered this problem (1.3) when $k = n$ and obtained some results via a gradient flow method. In particular, Tso raised the question of existence of solutions to (1.3) when F is bounded between two positive constants. It should be noted that unless F is constant, Oliker and Delanoë's results do not cover this case. In [16], Li addressed this question. Here Li used a topological degree theory method to study this problem. It should be noted that the advantage to the degree theory approach is that one does not need to show that the kernel on the linearized operator for (1.3) is trivial. However, in order to apply the degree theory argument, it is necessary that the function F be group invariant under a fixed-point free subgroup of $O(n + 1)$.

In [16], Li showed that if F is strictly monotone in one direction, then there does not exist a solution to (1.3). In particular, for $F(\vec{x}) = 10\pi + \tan^{-1} x_{n+1}$, there does not exist a solution to (1.3). Li addressed the issue of uniqueness of solutions to $W_k \equiv c > 0$ by using the method of moving planes, concluding that any embedded solution must be a sphere. Li showed as well that if F is bounded and group invariant under a fixed-point free subgroup $G \subset O(n+1)$, then there is a solution to (1.3). He also has some more general results for a wider class of functions F, which are a generalization of the condition of F being bounded between two constants, and G-invariant under a fixed-point free subgroup of $O(n+1)$.

Other results of interest are Guan, Lin and Ma [9] who addressed the issue of convexity in the general k case. Note that unlike the case of $k = n$, a solution to (1.3) may not be convex. See also results of Gerhardt, who addressed the issue of convexity of solutions of (1.3) as well as (1.3) in the setting of Riemannian manifolds and Lorentzian manifolds. Guan, Lin and Ma, Delanoë and Treibergs also considered (1.3) when F is homogeneous of degree $-k$. In this case, (1.3) may be viewed as a nonlinear eigenvalue problem.

1.3. **The Results in this Paper.** We consider this problem (1.3) when $n = k = 2$. We will assume that F is positive and satisfies certain growth assumptions near the origin and at infinity. F being positive insures that any hypersurface M with Gauss-Kronecker curvature $K(\vec{X}) = F(\vec{X})$ for $\vec{X} \in M$ is necessarily convex; this follows from a a well-known theorem of Hadamard. Moreover, we shall also assume that F is G-invariant, where $G \subset O(3)$ is a fixed-point free subgroup. Recall that a subgroup $G \subset O(n+1)$ is said to be fixed-point free if for any $\vec{x} \in \mathbb{S}^n$, there exists a $g \in G$ so that $g\vec{x} \neq x$. We note that this condition is necessary to establish our result via the degree theory argument established in [16]. Moreover, our result is an extension of that in [16], which proves this result in the case that $G = \mathbb{Z}_2$, as well as more general fixed-point free subgroups of the orthogonal group under stronger growth conditions. Our main result in this paper is the following theorem in $\mathbb{R}^3$:

Theorem 1.1. *Let* $F \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}, \mathbb{R}^+)$, $k \geq 2$, $\alpha \in (0,1)$, *satisfy* $F(g\vec{X}) = F(\vec{X})$ *for all* $\vec{X} \in \mathbb{R}^3 \setminus \{0\}$, $g \in G$, *where* $G \subset O(3)$ *is a fixed-point free subgroup of the orthogonal group. Suppose* F *also satisfies:* $\liminf_{|\vec{X}| \to \infty} F(\vec{X})|\vec{X}|^2 > 1$, $\limsup_{|\vec{X}| \to 0} F(\vec{X})|\vec{X}|^2 < 1$. *Then there exists* $\rho \in C^{k+2,\alpha}(\mathbb{S}^2, \mathbb{R}^+)$ *such that the surface* $M := \{\vec{X} = \rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$

is strictly convex and has Gauss-Kronecker curvature $K(\vec{X}) = F(\vec{X})$ *at* $\vec{X} = \rho(\vec{x})\vec{x}$*. Moreover,* $\rho(g\vec{x}) = \rho(\vec{x})$ *for all* $\vec{x} \in \mathbb{S}^2$*, and* $g \in G$*.*

We remark that the conditions on F in Theorem 1.1 are the weakest possible growth assumptions we may suppose. To see this, suppose $\vec{Y}, \vec{y} \in M$ satisfy $|\vec{Y}| = \sup_{\vec{z}\in M} |\vec{z}|$, $|\vec{y}| = \inf_{\vec{z}\in M} |\vec{z}|$, then we must have $K(\vec{Y}) \geq \frac{1}{|\vec{Y}|^2}$ and $K(\vec{y}) \leq \frac{1}{|\vec{y}|^2}$. Thus for any fixed $\varepsilon > 0$, the functions $F(\vec{X}) = \frac{1\pm\varepsilon}{|\vec{X}|^2}$ do not admit a solution hypersurface M with Gauss-Kronecker curvature $K(\vec{X}) = \frac{1\pm\varepsilon}{|\vec{X}|^2}$ for all $\vec{X} \in M$, for either choice of $\pm$. Moreover, in [16], Li showed that the function F which is strictly monotone in one direction does not admit a a hypersurface M which is a solution to $F(\vec{X}) = K(\vec{X})$ for all $\vec{X} \in M$. In particular, the function $F(\vec{X}) = 10\pi + \tan^{-1}(X_{n+1})$ does not admit a solution to $F(\vec{X}) = K(\vec{X})$. Hence, our group invariance assumption cannot be removed in general. We remark as well that a version of Theorem 1.1 is still an open problem in $\mathbb{R}^{n+1}$, with F satisfying the obvious analogous conditions, and $G \subset O(n+1)$ being fixed-point free.

Theorem 1.1 will be established by solving the following fully nonlinear elliptic equation of Monge-Ampere type for $u = \frac{1}{\rho}$, when $n = 2$.

$$(u^2 + |\nabla_{S^n} u|^2)^{-\frac{n+2}{2}} u^{n+2} \frac{\det(\nabla_{ij} u + u e_{ij})}{\det(e_{ij})} = F(\frac{1}{u(\vec{x})}\vec{x}) \tag{1.4}$$

for $u > 0$ in $\mathbb{S}^n$, where $e_{ij} =< e_i, e_j >$ is the metric for $\mathbb{S}^n$, $e_1, \cdots, e_n$ is a local frame and ∇ is covariant differentiation on $\mathbb{S}^n$ with respect to the usual metric on $\mathbb{S}^n$.

We remark that Theorem 1.1 will follow once we establish C^0 estimates for ρ. More specifically, by results of [4], [6], [13] and [18] we get higher regularity results. Then as in [16], we will use degree theory for group invariant functions to get the theorem. We shall establish the necessary C^0 estimates by establishing the following lemma.

Lemma 1.2. *Let* $0 < R_1 < R_2 < \infty$ *be constants, and suppose* $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$*. Suppose* $F \in C(\mathbb{R}^3 \setminus \{0\}, \mathbb{R}^+)$ *satisfies:*

$$F(\vec{X})|\vec{X}|^2 < 1 \quad \forall |\vec{X}| \in (0, R_1), \tag{1.5}$$

$$F(\vec{X})|\vec{X}|^2 > 1 \quad \forall |\vec{X}| \in (R_2, \infty), \tag{1.6}$$

$$\min_{\frac{1}{r}\leq s\leq r} \varphi(s) \leq F(\vec{X}) \leq \max_{\frac{1}{r}\leq s\leq r} \varphi(s) \quad 0 < \frac{1}{r} \leq |\vec{X}| \leq r < \infty, \tag{1.7}$$

$$F(g\vec{X}) = F(\vec{X}) \quad \forall \vec{X} \in \mathbb{R}^3 \setminus \{0\}, g \in G, \tag{1.8}$$

where $G \subset O(3)$ is a fixed-point free subgroup. Let $\rho \in C^2(\mathbb{S}^2, \mathbb{R}^+)$ satisfy $\rho(g\vec{x}) = \rho(\vec{x})$ for all $\vec{x} \in \mathbb{S}^2$, $g \in G$, and the Gauss-Kronecker curvature of $M = \{\vec{X} = \rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$ at $\vec{X} = \rho(\vec{x})\vec{x}$ is given by $K(\vec{X}) = F(\vec{X})$. Then for some positive $C = C(R_1, R_2, \varphi, G)$ we have

$$\max_{\mathbb{S}^2} \rho = \rho_{\max} \leq C, \tag{1.9}$$

$$\min_{\mathbb{S}^2} \rho = \rho_{\min} \geq \tfrac{1}{C}. \tag{1.10}$$

2. The Structure of the Proof of Lemma 1.2

We first note the following fact about fixed-point free subgroups of $O(3)$.

Proposition 2.1. *Let G be a fixed-point free subgroup of $O(3)$, then one of the following must hold*

(i) G contains $\mathrm{diag}\{-1,-1,-1\}$,

(ii) G contains an element g with complex conjugate eigenvalues $\alpha \pm i\beta$, $\beta \neq 0$,

(iii) there exists an orthonormal basis $\{e_1, e_2, e_3\}$ so that with respect to this basis, $H := \{\mathrm{diag}\{1,-1,-1\}, \mathrm{diag}\{-1,1,-1\}, \mathrm{diag}\{-1,-1,1\}, \mathrm{diag}\{1,1,1\}\} \subset G$.

Proof. Suppose that cases (i) and (ii) do not hold. Then $\mathrm{diag}\{-1,-1,-1\}$ is not in G and all elements of G have real eigenvalues. We first claim that there is an element in G with eigenvalues $-1,-1,1$, for if not, there are two distinct elements with eigenvalues $-1,1,1$, and their product has determinant 1, and we thus have an element with eigenvalues $-1,-1,1$ or 1 and two complex conjugate eigenvalues.

Thus $g = \mathrm{diag}\{-1,-1,1\} \in G$, when using an appropriate basis e_1, e_2, e_3 of $\mathbb{R}^3$. Since this fixes e_3, there is another distinct $\gamma \in G$, which has eigenvalues $-1,1,1$ or $-1,-1,1$. If γ has eigenvalues $-1,-1,1$, then with respect to the e_1, e_2, e_3 basis we have

$$\gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos 2\alpha & -\sin 2\alpha \\ 0 & -\sin 2\alpha & \cos 2\alpha \end{bmatrix},$$

for some $\alpha \in (0, \frac{\pi}{2}]$. By considering $g\gamma$, we find that $\alpha = \frac{\pi}{2}$, and this gives us (iii). If γ has eigenvalues $-1,1,1$, we get that in some basis we may write

$$g = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos 2\alpha & \sin 2\alpha \\ 0 & \sin 2\alpha & \cos 2\alpha \end{bmatrix}, \qquad \gamma = \begin{bmatrix} -\cos 2\beta & -\sin 2\beta & 0 \\ -\sin 2\beta & \cos 2\beta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for some $\alpha, |\beta| \in (0, \frac{\pi}{2}]$. We then consider the product $g\gamma$. -1 is clearly an eigenvalue for $g\gamma$ and the other eigenvalues must satisfy the equation $\lambda^2 - (\cos 2\alpha + \cos 2\beta - \cos 2\alpha \cos 2\beta + 1)\lambda + 1 = 0$. However, $|\cos 2\alpha + \cos 2\beta - \cos 2\alpha \cos 2\beta + 1| \leq 2$ on $[0, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. If this is less than 2, then $g\gamma$ has complex eigenvalues, and if this is equal to two we have either α or $\beta = 0$ which is not allowed, or $\text{diag}\{-1, -1, -1\} \in G$, which is a contradiction as well. □

We remark that if (i) holds, Theorem 1.1 follows from a result of [16]. Hence we may assume (ii) or (iii) holds in proving Lemma 1.2. If (ii) holds, then there exists $g \in G$ and an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ so that in this basis we have:

$$g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \tag{2.1}$$

for some $\theta \in (0, \pi)$ and choice of ± 1. Moreover, in the case where -1 is an eigenvalue of g, we distinguish between the cases $\theta = \frac{\pi}{2}$ and $\theta \neq \frac{\pi}{2}$. We also distinguish whether or not θ is a rational multiple of π. If θ is not a rational multiple of π, then it follows that $\rho|_{\{(x,y,z): z = \text{constant} \in (-1,1)\}}$ is constant. In the case where $\theta = \frac{m}{n}\pi$, $(m, n) = 1$, it follows that

$$\begin{bmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} & 0 \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G, \tag{2.2}$$

since we can always find an even k and some integer j so that $k\frac{m}{n}\pi = \frac{2}{n}\pi + 2j\pi$.

We also remark that in case (ii), if 1 is an eigenvalue of g, then we know there is another distinct element $\gamma \in G$. We find that γ must be of one of the following forms:

$$\gamma = \begin{bmatrix} \cos\phi & \cos\alpha \sin\phi & \sin\alpha \sin\phi \\ -\cos\alpha \sin\phi & \cos^2\alpha \cos\phi + \sin^2\alpha & \cos\alpha \sin\alpha(\cos\phi - 1) \\ -\sin\alpha \sin\phi & \cos\alpha \sin\alpha(\cos\phi - 1) & \sin^2\alpha \cos\phi + \cos^2\alpha \end{bmatrix} \tag{2.3}$$

for some $\alpha \in (0, \frac{\pi}{2}]$ and $\phi \in (0, \pi)$;

$$\gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos 2\alpha & -\sin 2\alpha \\ 0 & -\sin 2\alpha & \cos 2\alpha \end{bmatrix} \tag{2.4}$$

for some $\alpha \in (0, \frac{\pi}{2}]$;

$$\gamma = \begin{bmatrix} -\cos^2\beta + \sin^2\beta & -2\cos\beta\sin\beta\cos\alpha & -2\cos\beta\sin\beta\sin\alpha \\ -2\cos\beta\sin\beta\cos\alpha & 1-2\sin^2\beta\cos^2\alpha & -2\sin^2\beta\cos\alpha\sin\alpha \\ -2\cos\beta\sin\beta\sin\alpha & -2\sin^2\beta\cos\alpha\sin\alpha & 1-2\sin^2\beta\sin^2\alpha \end{bmatrix}, \quad (2.5)$$

where $\alpha \in (0, \frac{\pi}{2}]$, $|\beta| \in (0, \frac{\pi}{2}]$. We note that case (2.3) corresponds to a case where γ has complex eigenvalues, (2.4) corresponds to the case where γ has eigenvalues $-1, -1, 1$ and (2.5) is the case where γ has eigenvalues $-1, 1, 1$.

3. Some Initial Estimates

Let ρ be a solution to (1.4). We will now establish some a priori estimates for ρ.

At this point we recall a result of [16] (Lemmas 2.1 and 2.2), which we shall give a proof of here as well.

Lemma 3.1. *Under the assumptions of Lemma* 1.2 *we have*

$$\min_{\mathbb{S}^2} \rho = \rho_{\min} \leq R_2, \quad (3.1)$$

$$\max_{\mathbb{S}^2} \rho = \rho_{\max} \geq R_1. \quad (3.2)$$

Proof. The inner unit normal to M at $\vec{X} = \rho(\vec{x})\vec{x}$ is given by

$$\frac{\nabla_{\mathbb{S}^n}\rho(\vec{x}) - \rho(\vec{x})\vec{x}}{\sqrt{|\nabla_{\mathbb{S}^n}\rho(\vec{x})|^2 + \rho^2(\vec{x})}}.$$

Let $\vec{x}_{\min}, \vec{x}_{\max} \in \mathbb{S}^n$ be such that $\rho_{\min} = \rho(\vec{x}_{\min})$ and $\rho_{\max} = \rho(\vec{x}_{\max})$. By comparing the Gauss curvature of M at $\rho_{\min}\vec{x}_{\min}$ with a ball of radius $\rho_{\min}$ and the Gauss curvature of M at $\rho_{\max}\vec{x}_{\max}$ with a ball of radius $\rho_{\max}$, the result then follows. □

Lemma 3.2. *Suppose* $\max_{\mathbb{S}^2} \rho \geq R_1$, $\min_{\mathbb{S}^2} \rho \leq R_2$ *and* $\min_{\mathbb{S}^2} \rho \geq C \max_{S^2} \rho$. *Then there exists* $1 \leq \tilde{C} < \infty$ *so that* $\frac{1}{\tilde{C}} \leq \rho \leq \tilde{C}$ *on* $\mathbb{S}^2$.

Lemma 3.3. *Suppose that with respect to some orthonormal basis* $\{e_1, e_2, e_3\}$ *of* $\mathbb{R}^3$, *which gives rise to a coordinate system* (x, y, z), *we have*

$$g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}.$$

If -1 *is an eigenvalue of* g, *we shall suppose* $\theta \neq \frac{\pi}{2}$ *holds as well. Then for any* $P \in S^2$ *we have*

$$\rho|_{\{(x,y,z): z=z_P\}} \geq \frac{\rho(P)}{2}, \quad (3.3)$$

where $P = z_P e_1 + y_P e_2 + z_P e_3 = (x_P, y_P, z_P)$.

Proof. We need only consider the case where $\theta = \frac{m}{n}\pi$. By (2.2), the result follows using convexity. □

Lemma 3.4. *Suppose*

$$g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \in G,$$

for some $\theta \neq \frac{\pi}{2}$. *Let* $P \in \mathbb{S}^2$ *be a minimum point of* ρ. *Let* $P = x_P e_1 + y_P e_2 + z_P e_3$. *Then for* $z_P \notin \{0, \pm 1\}$ *we have*

$$M \subset \{(x,y,z) : |z| \leq \frac{\min_{\mathbb{S}^2} \rho}{|z_P|}, \sqrt{x^2+y^2} \leq 2\frac{\min_{\mathbb{S}^2} \rho}{\sqrt{1-z_P^2}}\}. \tag{3.4}$$

For $|z_P| = 1$, *we have*

$$M \subset \{(x,y,z) : |z| \leq \min_{\mathbb{S}^2} \rho\}, \tag{3.5}$$

and for $z_P = 0$ *we have*

$$M \subset \{(x,y,z) : z \in \mathbb{R}, \sqrt{x^2+y^2} \leq 2 \min_{\mathbb{S}^2} \rho\}. \tag{3.6}$$

Proof. We know that $\tilde{M} \subset \{\vec{X} \in \mathbb{R}^3 : \vec{X} \cdot \gamma P \leq \min_{\mathbb{S}^2} \rho, \forall \gamma \in G\}$. Let $\vec{X} =< \vec{X}, \vec{X}' > \vec{X}' + < \vec{X}, e_3 > e_3$, $\|\vec{X}'\| = 1$, $< \vec{X}, \vec{X}' > \geq 0$, $< \vec{X}', e_3 >= 0$ and $P =< P, P' > P' + < P, e_3 > e_3$, $\|P'\| = 1$, $< P, P' > \geq 0$, $< P', e_3 >= 0$. Then

$$\min_{\mathbb{S}^2} \rho \geq \vec{X} \cdot \gamma P$$

$$=< \vec{X}, \vec{X}' >< P, P' >< \vec{X}', \gamma P' > + < \vec{X}, e_3 >< P, e_3 >< \gamma e_3, e_3 > .$$

If $P = e_3$, then $< \vec{X}, e_3 >< e_3, \gamma e_3 > \leq \min_{\mathbb{S}^2} \rho$. Thus we have

$$M \subset \{(x,y,z) : |z| \leq \min_{\mathbb{S}^2} \rho\}.$$

If $< P, e_3 >= 0$, then $< \vec{X}, \vec{X}' >< \vec{X}', \gamma P > \leq \min_{\mathbb{S}^2} \rho$. We can choose γ so that $< \vec{X}', \gamma P > \geq \frac{1}{2}$, and thus we get

$$M \subset \{(x,y,z) : z \in \mathbb{R}, \sqrt{x^2+y^2} \leq 2 \min_{\mathbb{S}^2} \rho\}.$$

In the case where $< P, P' >, < P, e_3 > \neq 0$, we first require $< \vec{X}, e_3 >< \gamma e_3, e_3 >= | < \vec{X}, e_3 > |$, by picking γ appropriately. We then get $< \vec{X}, \vec{X}' >< \vec{X}', \gamma P' >< P, P' > \leq \min_{\mathbb{S}^2} \rho$. By appropriate choice of γ we also require $< \vec{X}', \gamma P' > \geq \frac{1}{2}$. Thus we get $< \vec{X}, \vec{X}' > \leq 2\frac{\min_{\mathbb{S}^2} \rho}{< P, P' >}$.

Next, we choose γ such that $< \vec{X}', \gamma P' >> 0$ and $< \vec{X}, e_3 >< \gamma e_3, e_3 > = | < \vec{X}, e_3 > |$. Thus, we get $| < \vec{X}, e_3 > | \leq \frac{\min_{\mathbb{S}^2} \rho}{<P, e_3>}$. Hence,

$$M \subset \{(x, y, z) : |z| \leq \frac{\min_{\mathbb{S}^2} \rho}{|z_P|}, \sqrt{x^2 + y^2} \leq 2 \frac{\min_{\mathbb{S}^2} \rho}{\sqrt{1 - z_P^2}}\}. \qquad \square$$

4. Proof of Lemma 1.2

Proposition 4.1. *Suppose $M_i = \{\rho_i(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$ is a family of hypersurfaces with Gauss curvature K_i satisfying (1.5)-(1.8) for some $0 < R_1 < R_2 < \infty$, $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$ and some fixed-point free $G \subset O(3)$. Suppose*

$$\sup\{|\vec{X}| : \vec{X} = (x, y, z) \in M_i, 1 \leq z \leq 2\} = O(1).$$

Let $A_i = \max_{\mathbb{S}^2} \rho_i$. Suppose there exists $0 < \tilde{C} < 1$, independent of i, so that for $B_i := \tilde{C} A_i$ we have $\{(0, 0, tB_i) : t \in [-1, 1]\} \subset \tilde{M}_i$. Then (1.9) holds for $\{\rho_i\}$, for some $C = C(R_1, R_2, \varphi, \tilde{C})$.

Proof. Suppose the contrary, then we define

$$\bar{\delta}_i := \sup\{r > 0 : (x, y, 2) \in M_i, x^2 + y^2 < r^2\}.$$

We know $\bar{\delta}_i = O(1)$. By convexity of M_i, we know that $\{(x, y, 1) : x^2 + y^2 \leq \delta_i^2\} \subset \tilde{M}_i$, where $\delta_i := \bar{\delta}_i(1 - \frac{1}{B_i + 2})$ (use similar triangles to see this). Also, by (1.7) of Lemma 1.2 we have

$$\inf\{K_i(\vec{X}) : \vec{X} = (x, y, z) \in M_i, 1 \leq z \leq 2\} \geq C_0 > 0, \qquad (4.1)$$

with C_0 independent of i.

We consider the comparison surface

$$S_i(t) = \{(x, y, z) : 1 \leq z \leq 2, x^2 + y^2 = \delta_i^2(1 + t \sin[(z - 1)\pi])\}.$$

We know that the Gauss-Kronecker curvature of $S_i(t)$ is $O(t)$. It follows from (4.1) that there exists $t_0 > 0$ such that $S_i(t)$ lies inside M_i for $0 \leq t \leq t_0$. Thus, it follows that $\{(x, y, \frac{3}{2}) : x^2 + y^2 < (1 + t_0)\delta_i^2\} \subset \tilde{M}_i$. Then by convexity, we have that

$$\{(x, y, z) : \tfrac{3}{2} \leq z \leq B_i, x^2 + y^2 \leq [\sqrt{1 + t_0} \delta_i \frac{B_i - z}{B_i - \frac{3}{2}}]^2\} \subset \tilde{M}_i.$$

Hence, we get that

$$\{\vec{X} = (x, y, 2) : x^2 + y^2 \leq [\sqrt{1 + t_0} \delta_i \frac{B_i - 2}{B_i - \frac{3}{2}}]^2\} \subset \tilde{M}_i.$$

Thus, by the definition of $\bar{\delta}_i$ we have

$$\sqrt{1+t_0}\frac{B_i-2}{B_i-\frac{3}{2}}\delta_i \leq \bar{\delta}_i = (1-\frac{1}{B_i+2})^{-1}\delta_i,$$

and hence

$$\sqrt{1+t_0}(\frac{B_i-2}{B_i-\frac{3}{2}})(1-\frac{1}{B_i+2}) \leq 1.$$

This is a contradiction if $i >> 1$. □

Proposition 4.2. *Suppose $M_i = \{\rho_i(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$ is a family of hypersurfaces with Gauss curvature K_i satisfying* (1.5)-(1.8) *for some $0 < R_1 < R_2 < \infty$, $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$ and some fixed-point free $G \subset O(3)$. Let $A_i = \max_{\mathbb{S}^2} \rho_i$. Suppose there exists $0 < \tilde{C} < 1$, independent of i, so that for $B_i := \tilde{C}A_i$ we have $\{(x, y, z) : z = 0, \sqrt{x^2+y^2} \leq B_i\} \subset \tilde{M}_i$. Then* (1.9) *holds for $\{\rho_i\}$, for some $C = C(R_1, R_2, \varphi, \tilde{C})$.*

Proof. Suppose the contrary. Since $B_i \to \infty$, we may assume $B_i > 6$ for all i, which implies that

$$\{(x, y, z) : (x-a)^2 + (y-b)^2 \leq 1, a^2 + b^2 = 4, z = 0\} \subset \tilde{M}_i,$$

for any choice of (a, b) satisfying $a^2 + b^2 = 4$. In particular,

$$D := \{(x, y, z) : x^2 + (y-2)^2 \leq 1, z = 0\} \subset \tilde{M}_i.$$

Consider

$$S := \{(x, y, z) : x^2 + (y-2)^2 + z^2 = 1, z \geq 0\}.$$

We may deform D to S in a continuous manner by a family of surfaces $S(t)$ so that $S(0) = D$, $S(1) = S$ and so that any point $P \in S(t) \setminus \partial S(t)$ has Gauss-Kronecker curvature $K(P) = t^2$. Thus the Gauss-Kronecker curvature is $O(t)$ as $t \to 0$.

Moreover, by our bound (1.7) of Lemma 1.2, we see that there exists $t_0 > 0$ so that, for $0 \leq t \leq t_0$, $S(t)$ lies inside $\tilde{M}_i$. Also, $\tilde{M}_i$ contains the reflection of $S(t)$ about the xy-plane, for any $t \in [0, t_0]$. In addition, the choice of $a = 0$ and $b = 2$ was arbitrary, thus we get that

$$\{(x, y, z) : -\sqrt{t_0} \leq z \leq \sqrt{t_0}, x^2 + y^2 \leq 4\} \subset \tilde{M}_i.$$

Hence, $\min_{\mathbb{S}^2} \rho_i \geq \sqrt{t_0}$.

Now, for $t \in [0, 1]$, consider the following comparison surfaces $\Sigma_i(0) = \{(x, y, 0) : x^2 + y^2 \leq \frac{B_i^2}{4}\}$, and for $t \in (0, 1]$ $\Sigma_i(t) := \{(x, y, z) : x^2 + y^2 \leq \frac{B_i^2}{4}$,

$$z = \sqrt{(\frac{B_i}{2} + \frac{1}{t} - 1)^2 - (x^2+y^2)} - \sqrt{(\frac{B_i}{2} + \frac{1}{t} - 1)^2 - \frac{B_i^2}{4}}\}.$$

The $\Sigma_i(t)$ are spherical caps of spheres with radii $\frac{B_i}{2}+\frac{1}{t}-1$, passing through the curve $\{(x,y,0): x^2+y^2=\frac{B_i^2}{4}\}$ and having centers

$$(0,0,-\sqrt{(\frac{B_i}{2}+\frac{1}{t}-1)^2-(\frac{B_i}{2})^2}).$$

Hence, the Gauss-Kronecker curvature of any interior point $\vec{X}\in\Sigma_i(t)$ has value $\frac{1}{(\frac{B_i}{2}+\frac{1}{t}-1)^2}$. Clearly, $\Sigma_i(0)\cap M_i=\phi$. Let $t_i^*\in(0,1]$ be the first time (if any) that there exists $\vec{X}_i\in\Sigma_i(t)\cap M_i$. Thus for $t<t_i^*$ we have $\Sigma_i(t)\cap M_i=\phi$. Moreover, at any such $\vec{X}_i$ we have $\vec{X}_i\in\Sigma_i(t_i^*)\setminus\partial\Sigma_i(t_i^*)$, because for each fixed i, $\partial\Sigma_i(0)=\partial\Sigma_i(t)$ for all $t\in[0,1]$. Therefore,

$$K_i(\vec{X}_i)\leq\frac{1}{(\frac{B_i}{2}+\frac{1}{t_i^*}-1)^2}.$$

However,

$$|\vec{X}_i|^2\leq(\frac{B_i}{2}+\frac{1}{t_i^*}-1)^2.$$

Thus, $K_i(\vec{X}_i)|\vec{X}_i|^2\leq 1$, and by (1.6) of Lemma 1.2 $|\vec{X}_i|\leq R_2$. Thus, $|\vec{X}_i|\in[\sqrt{t_0},R_2]$ and $K_i(\vec{X}_i)\leq\frac{1}{(\frac{B_i}{2}+\frac{1}{t_i^*}-1)^2}$. This is a contradiction to

$$\min_{\sqrt{t_0}\leq s\leq R_2}\varphi(s)\leq K_i(\vec{X}_i)\leq\max_{\sqrt{t_0}\leq s\leq R_2}\varphi(s)$$

if $i>>1$. □

Proposition 4.3. *Suppose $M_i=\{\rho_i(\vec{x})\vec{x}:\vec{x}\in\mathbb{S}^2\}$ is a family of hypersurfaces with Gauss curvature K_i satisfying* (1.5)-(1.8) *for some $0<R_1<R_2<\infty$, $\varphi\in C(\mathbb{R}^+,\mathbb{R}^+)$ and some fixed-point free $G\subset O(3)$. Let $\varepsilon_i=\min_{\mathbb{S}^2}\rho_i$. Suppose there exists $0<\tilde{C},\hat{C}<\infty$, independent of i, so that $M_i\subset\{(x,y,z):|z|\leq\tilde{C}\varepsilon_i\}$, and $\max_{\mathbb{S}^2}\rho_i\leq\hat{C}$. Then* (1.10) *holds for $\{\rho_i\}$, for some $C=C(R_1,R_2,\varphi,\tilde{C},\hat{C})$.*

Proof. By changing our coordinate system, we may also suppose that a maximum point of ρ_i lies in the xz-plane with $x\geq 0$. If $\hat{X}_i=(x^i,0,z^i)$ is a such a maximum point, where $|\hat{X}_i|\geq R_1$, then $(x^i)^2\geq R_1^2-(z^i)^2\geq R_1^2-\tilde{C}^2\varepsilon_i^2$. Thus we have $x^i\geq\frac{3}{4}R_1$ for $i>>1$.

Next, we shall construct a family of comparison surfaces. Consider the curve $C^i:=\{(x,y,z): y=0, x\geq\hat{C}+1, (x-\hat{C}-1)^2+z^2=(2\tilde{C}\varepsilon_i)^2\}$. Let S_i be the surface obtained by rotating C^i about the z-axis and keeping only the portion in $\{(x,y,z) : x \geq 0\}$. Let $S_i(t)$ be the surface obtained by translating S_i in the direction $(-t,0,0)$.

For $t=0$ we know $S_i(0)\cap M_i=\phi$; this follows from the construction. Let $t_i^*>0$ denote the first time when $S_i(t)\cap M_i$ is non-empty, and let $\vec{X}_i \in S_i(t_i^*)\cap M_i$. Clearly, we know $|\vec{X}_i| \geq \frac{R_1}{2}$ if $i >> 1$. $K_i(\vec{X}_i)$, the Gauss-Kronecker curvature of M_i at $\vec{X}_i$, is greater than the Gauss-Kronecker curvature of $S_i(t_i^*)$ at $\vec{X}_i$. This in turn is greater than or equal to $\frac{1}{Constant\cdot\varepsilon_i}$, which tends to infinity as $i\to\infty$. This gives the desired contradiction via assumption (1.7) of Lemma 1.2, which tells us

$$\min_{\frac{R_1}{2}\leq s\leq \hat{C}} \varphi(s) \leq K_i(\vec{X}_i) \leq \max_{\frac{R_1}{2}\leq s\leq \hat{C}} \varphi(s). \qquad \square$$

4.1. **Case where G has an element with eigenvalues -1, $\cos\theta \pm i\sin\theta$, $\theta\neq\frac{\pi}{2}$.** Suppose (1.9) of Lemma 1.2 does not hold. Then there exists R_1, R_2, φ, G, with $g=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & -1\end{bmatrix}\in G$, $\theta\neq\frac{\pi}{2}$, and families $\{K_i\}_{i=1}^\infty$, $\{\rho_i\}_{i=1}^\infty$ such that (1.5)-(1.8) in the statement of Lemma 1.2 hold, but $A_i := \max_{\mathbb{S}^2}\rho_i\to\infty$ as $i\to\infty$, where K_i is the Gauss-Kronecker curvature of $M_i:=\{\rho_i(\vec{x})\vec{x} : \vec{x}\in\mathbb{S}^2\}$. There is an $\vec{x}_i\in\mathbb{S}^2\cap\{(x,y,z) : z\geq 0\}$ so that $\rho_i(\vec{x}_i)=\max_{\mathbb{S}^2}\rho_i$. Fix $0<\varepsilon<<1$. We define

$$\theta_i = \cos^{-1} < \vec{x}_i, (0,0,1) >,$$

and we consider whether $\theta_i\in[0,\varepsilon)$, $\theta_i\in[\varepsilon,\frac{\pi}{2}-\varepsilon]$, or $\theta_i\in(\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}]$. Without loss of generality, by otherwise taking a subsequence and re-indexing, we may assume that for all $i\in\mathbb{N}$ only one of the three possible cases happens.

First we assume that for all i, $\theta_i\in[\varepsilon,\frac{\pi}{2}-\varepsilon]$. By Lemma 3.3 and convexity, we know that $\{(x,y,z) : |z|\leq A_i\cos\theta_i, x^2+y^2\leq\frac{A_i^2}{4}\sin^2\theta_i\}\subset\tilde{M}_i$. Then using our bounds on θ_i, we get $\{\vec{X}\in\mathbb{R}^3 : |\vec{X}|\leq\frac{A_i}{2}\sin\varepsilon\}\subset\tilde{M}_i$. This gives a contradiction to $\min_{\mathbb{S}^3}\rho_i\leq R_2$.

Now we assume $\theta_i\in[0,\varepsilon)$ for all i. As above, we see that $\{(x,y,z) : |z|\leq A_i\cos\theta_i, x^2+y^2\leq\frac{A_i^2}{4}\sin^2\theta_i\}\subset\tilde{M}_i$, and thus we have $\{\vec{X}\in\mathbb{R}^3 : |\vec{X}|\leq\frac{A_i}{2}\sin\theta_i\}\subset\tilde{M}_i$. Thus $R_2\geq\min_{\mathbb{S}^2}\rho_i\geq\frac{A_i}{2}\sin\theta_i$, and hence we have $\theta_i\to 0$ as $i\to\infty$.

Next we claim that $\sup\{|\vec{X}| : \vec{X}=(x,y,z)\in M_i, z\in[1,2]\}=O(1)$. For suppose not, then there is $P_i\in\mathbb{S}^2$ so that $\rho_i(P_i)\to\infty$ and $z_i =< \rho_i(P_i)P_i,(0,0,1) >\in[1,2]$. Then Lemma 3.3 and convexity yield $\{(x,y,z) :$

$|z| \leq A_i \cos\varepsilon, x^2 + y^2 \leq R_i^2(z)\} \subset \tilde{M}_i$, where

$$R_i(z) = \frac{A_i \cos\varepsilon - z}{2(A_i \cos\varepsilon - z_i)} \sqrt{\rho_i^2(P_i) - z_i^2}$$

if $z_i \leq z \leq A_i \cos\varepsilon$ and

$$R_i(z) = \frac{A_i \cos\varepsilon + z}{2(A_i \cos\varepsilon + z_i)} \sqrt{\rho_i^2(P_i) - z_i^2}$$

if $-A_i \cos\varepsilon \leq z \leq z_i$. Thus we obtain that for $i >> 1$ we have $\{(x, y, z) : |z| \leq \min\{\rho_i(P_i), A_i \cos\varepsilon, \sqrt{x^2 + y^2} \leq \frac{1}{4}(\min\{\rho_i(P_i), A_i \cos\varepsilon\} - |z|)\} \subset \tilde{M}_i$. It follows that for $i >> 1$ we have that

$$\{\vec{X} \in \mathbb{R}^3 : |\vec{X}| \leq \frac{1}{\sqrt{17}} \min\{A_i \cos\varepsilon, \rho_i(P_i)\}\} \subset \tilde{M}_i.$$

This contradicts the fact that $\min_{\mathbb{S}^2} \rho_i \leq R_2$. We thus obtain that $\sup\{|\vec{X}| : \vec{X} = (x, y, z) \in M_i, z \in [1, 2]\} = O(1)$. We then apply Proposition 4.1 to get the desired result.

Lastly we suppose that, for all i, $\theta_i \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}]$. In this case we have $\{(x, y, z) : z = 0, x^2 + y^2 \leq \frac{1}{4} A_i^2 \cos^2\varepsilon\} \subset \tilde{M}_i$. Hence, the result follows from Proposition 4.2. Thus we have established (1.9) in this case.

Now we will establish (1.10). Suppose the contrary, then there exists R_1, R_2, φ, G, $g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \in G$, with $\theta \neq \frac{\pi}{2}$, and families $\{K_i\}_{i=1}^{\infty}$, $\{\rho_i\}_{i=1}^{\infty}$ so that (1.5)-(1.8) of Lemma 1.2 hold, but $\varepsilon_i := \min_{\mathbb{S}^2} \rho_i \to 0$ as $i \to \infty$, where K_i is the Gauss-Kronecker curvature of $M_i := \{\rho_i(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$.

Fix $0 < \varepsilon << 1$. Let $\vec{x}_i \in \mathbb{S}^2 \cap \{(x, y, z) : z \geq 0\}$ be such that $\rho_i(\vec{x}_i) = \min_{\mathbb{S}^2} \rho_i$ and let $\varphi_i := \cos^{-1} < \vec{x}_i, (0, 0, 1) >$. Without loss of generality, we suppose for all $i \in \mathbb{N}$ one of the following: $\varphi_i \in [0, \varepsilon)$, $\varphi_i \in [\varepsilon, \frac{\pi}{2} - \varepsilon]$, or $\varphi_i \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}]$.

First we suppose that $\varphi_i \in [\varepsilon, \frac{\pi}{2} - \varepsilon]$. By Lemma 3.4, we have $M_i \subset \{(x, y, z) : |z| \leq \frac{\varepsilon_i}{\cos\varphi_i}, x^2 + y^2 \leq [2\frac{\varepsilon_i}{\sin\varphi_i}]^2\}$. Now, $\frac{\varepsilon_i}{\sin\varphi_i} \leq \frac{\varepsilon_i}{\sin\varepsilon} \to 0$ as $i \to \infty$, and $\frac{\varepsilon_i}{\cos\varphi_i} \leq \frac{\varepsilon_i}{\sin\varepsilon} \to 0$ as $i \to \infty$. This is a contradiction to $\max_{\mathbb{S}^2} \rho_i \geq R_1$. Hence $\varphi_i \in [\varepsilon, \frac{\pi}{2} - \varepsilon]$ cannot happen for infinitely many $i \in \mathbb{N}$.

Nest we suppose that $\varphi_i \in [0, \varepsilon)$ for all $i \in \mathbb{N}$. Then by Lemma 3.4, we have $M_i \subset \{(x, y, z) : |z| \leq \frac{\varepsilon_i}{\cos\varphi_i}, x^2 + y^2 \leq [2\frac{\varepsilon_i}{\sin\varphi_i}]^2\}$. Thus, we get $\varphi_i \to 0$. Moreover, for i large we have $\tilde{M}_i \subset \{(x, y, z) : |z| \leq 2\varepsilon_i\}$. We then apply Proposition 4.3 to establish (1.10).

Lastly, we assume $\varphi_i \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}]$ for all i. By Lemma 3.4 we have $M_i \subset \{(x,y,z) : |z| \leq \frac{\varepsilon_i}{\cos\varphi_i}, x^2+y^2 \leq [2\frac{\varepsilon_i}{\sin\varphi_i}]^2\}$, and thus $\varphi_i \to \frac{\pi}{2}$. Thus for large i we have $M_i \subset \{(x,y,z) : |x| \leq 4\varepsilon_i\}$. Hence the result follows by applying Proposition 4.3.

4.2. **Case where G has an element with eigenvalues -1, $\pm i$.** In this case, we may assume $g = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in G$. Suppose $P = (x_P, y_P, z_P) \in \mathbb{S}^2$ is a maximum point for ρ. By considering all points of the form $\rho(g^kP)g^kP \in M$, $k \in \mathbb{Z}$, then convexity yields

$$\{(x,y,z) : |z| \leq \frac{z_P}{2}\rho_{\max}, \sqrt{x^2+y^2} \leq \frac{\sqrt{1-z_P^2}}{8}\rho_{\max}\} \subset \tilde{M},$$

$$\{\vec{x} \in \mathbb{R}^3 : |\vec{x}| \leq \frac{\sqrt{1-z_P^2}|z_P|}{\sqrt{3z_P^2+1}}\rho_{\max}\} \subset \tilde{M}. \tag{4.2}$$

Let $Q = (x_Q, y_Q, z_Q) \in \mathbb{S}^2$ be a minimum point for ρ. The tangent plane to M at any point of the form $\rho(g^kQ)g^kQ$, $k \in \mathbb{Z}$ is a supporting hyperplane for M. The region enclosed by these planes contains M, and thus we get

$$M \subset \{(x,y,z) : |z| \leq \frac{\rho_{\min}}{|z_Q|}, \sqrt{x^2+y^2} \leq \frac{2\rho_{\min}}{\sqrt{1-z_Q^2}}\},$$

$$M \subset \{\vec{x} \in \mathbb{R}^3 : |\vec{x}|^2 \leq \rho_{\min}^2(\frac{4}{1-z_Q^2} + \frac{1}{z_Q^2})\}. \tag{4.3}$$

Suppose (1.9) of Lemma 1.2 does not hold. Then there exists R_1, R_2, φ, G, with $g = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in G$, and families $\{K_i\}_{i=1}^{\infty}$, $\{\rho_i\}_{i=1}^{\infty}$ such that (1.5)-(1.8) in the statement of Lemma 1.2 hold, but $A_i := \max_{\mathbb{S}^2} \rho_i \to \infty$ as $i \to \infty$, where K_i is the Gauss-Kronecker curvature of $M_i := \{\rho_i(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$. By otherwise changing our coordinate system, we may suppose $\vec{x}_i = (0, \sin\phi_i, \cos\phi_i)$ is such that $\rho_i(\vec{x}_i) = \max_{\mathbb{S}^2} \rho_i$. Fix $0 < \varepsilon << 1$. We consider whether $\phi_i \in [0, \varepsilon)$, $\phi_i \in [\varepsilon, \frac{\pi}{2} - \varepsilon]$, or $\phi_i \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}]$. Without loss of generality, by otherwise taking a subsequence and re-indexing, we may assume that for all $i \in \mathbb{N}$ only one of the three possible cases happens.

First, we suppose $\phi_i \in [\varepsilon, \frac{\pi}{2} - \varepsilon]$, then $\{\vec{x} \in \mathbb{R}^3 : |\vec{x}| \leq \frac{\sin^2\varepsilon}{\sqrt{3\cos^2\varepsilon + 1}}A_i\} \subset \tilde{M}_i$, which is a contradiction to $\rho_{\min} \leq R_2$ for $i >> 1$.

Next we suppose $\phi_i \in [0,\varepsilon)$ for all i. Since $R_2 \geq \min_{\mathbb{S}^2} \rho_i \geq A_i \frac{\sin\phi_i \cos\phi_i}{\sqrt{3\cos^2\phi_i+1}} \geq \frac{1}{2}A_i \sin\phi_i \cos\varepsilon$, $\sin\phi_i \leq \frac{2R_2}{A_i\cos\varepsilon} \to 0$ as $i \to \infty$. Thus $\phi_i \to 0$ as $i \to \infty$. Now $\{(0,0,t) : |t| \leq A_i\cos\varepsilon\} \subset \tilde{M}_i$, and we claim $\sup\{|\vec{x}| : \vec{x} = (x,y,z) \in M_i, 1 \leq z \leq 2\} = O(1)$. If this did not hold, then by restricting to a subsequence if necessary, we see that $\tilde{M}_i$ contains a ball of radius which $\to \infty$ as $i \to \infty$, and this contradicts $\min_{\mathbb{S}^2} \rho_i \leq R_2$. We get (1.9) by applying Proposition 4.1. Lastly, we suppose $\phi_i \in (\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}]$. Now $A_i \frac{\sin\phi_i\cos\phi_i}{\sqrt{3\cos^2\phi_i+1}} \geq \frac{A_i}{2}\cos\phi_i\cos\varepsilon$, and thus $R_2 \geq \min_{\mathbb{S}^2}\rho_i \geq \frac{A_i}{2}\cos\phi_i\cos\varepsilon$. Thus $\phi_i \to \frac{\pi}{2}$. Now $\{(x,y,z) : z = 0, \sqrt{x^2+y^2} \leq \frac{A_i}{8}\sin\phi_i\} \subset \tilde{M}_i$, and the result follows from Proposition 4.2.

Now we will establish (1.10). Suppose the contrary, then there exists R_1, R_2, φ, G, $g = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in G$, and families $\{K_i\}_{i=1}^{\infty}$, $\{\rho_i\}_{i=1}^{\infty}$ so that (1.5)-(1.8) of Lemma 1.2 hold, but $\varepsilon_i := \min_{\mathbb{S}^2}\rho_i \to 0 \quad \text{as} \quad i \to \infty$, where K_i is the Gauss-Kronecker curvature of $M_i := \{\rho_i(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$.

Fix $0 < \varepsilon << 1$. By otherwise changing our coordinate system, we may suppose $\vec{x}_i = (0, \sin\varphi_i, \cos\varphi_i)$ is such that $\rho_i(\vec{x}_i) = \min_{\mathbb{S}^2}\rho_i$. Without loss of generality, we suppose for all $i \in \mathbb{N}$ one of the following: $\varphi_i \in [0,\varepsilon)$, $\varphi_i \in [\varepsilon, \frac{\pi}{2}-\varepsilon]$, or $\varphi_i \in (\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}]$.

First we suppose $\varphi_i \in [\varepsilon, \frac{\pi}{2}-\varepsilon]$. Then $\tilde{M}_i \subset \{\vec{x} : |\vec{x}| \leq \varepsilon_i \frac{\sqrt{5}}{\sin\varepsilon}\}$. This is a contradiction to $\max_{\mathbb{S}^2}\rho_i \geq R_1$ for $i >> 1$. Next we suppose $\varphi_i \in [0,\varepsilon)$. In this case we use $M_i \subset \{(x,y,z) : |z| \leq \frac{\varepsilon_i}{\cos\varepsilon}\}$. Lastly, if $\varphi_i \in (\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}]$ we get $M_i \subset \{(x,y,z) : |x| \leq 2\frac{\varepsilon_i}{\cos\varepsilon}\}$. The result then follows from Proposition 4.3.

4.3. **Case where $H \subset G$.** Let $P = (x_P, y_P, z_P) \in \mathbb{S}^2$ be a maximum point for ρ. Let $\{P^1, P^2, P^3, P^4\} = \{hP : h \in H\}$. Let Π_j be the plane determined by $\{\rho_{\max}P^1, \cdots, \rho_{\max}P^4\} \setminus \{\rho_{\max}P^j\}$. The region defined by the planes $\Pi_1, \cdots, \Pi_4$ contains the ball $\{\vec{x} \in \mathbb{R}^3 : |\vec{x}| \leq \frac{|x_P y_P z_P|}{\sqrt{z_P^2(1-z_P^2)+x_P^2y_P^2}}\rho_{\max}\}$. Therefore,

$$\{\vec{x} \in \mathbb{R}^3 : |\vec{x}| \leq \frac{|x_P y_P z_P|}{\sqrt{z_P^2(1-z_P^2)+x_P^2y_P^2}}\rho_{\max}\} \subset \tilde{M}.$$

More specifically,

$$\{(x,y,z) : |x| \leq |x_P|\rho_{\max}, |y| \leq |\frac{y_P}{x_P}|(|x_P|\rho_{\max} - |x|), z = 0\} \subset \tilde{M},$$

$$\{(x,y,z): |x| \le |x_P|\rho_{\max}, |z| \le \frac{1-z_P^2}{|z_P||x_P|}(|x_P|\rho_{\max} - |x|), y=0\} \subset \tilde{M},$$

$$\{(x,y,z): |y| \le |y_P|\rho_{\max}, |z| \le \frac{1-z_P^2}{|z_P||y_P|}(|y_P|\rho_{\max} - |y|), x=0\} \subset \tilde{M}.$$

Suppose (1.9) of Lemma 1.2 does not hold. Then there exists R_1, R_2, φ, G, and families $\{K_i\}_{i=1}^{\infty}$, $\{\rho_i\}_{i=1}^{\infty}$ such that (1.5)-(1.8) in the statement of Lemma 1.2 hold, but $A_i := \max_{\mathbb{S}^2} \rho_i \to \infty$ as $i \to \infty$, where K_i is the Gauss-Kronecker curvature of $M_i := \{\rho_i(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$. There is an $\vec{x}_i = (\cos\alpha_i \cos\beta_i, \sin\alpha_i \cos\beta_i, \sin\beta_i)$, with $\alpha_i, |\beta_i| \in [0, \frac{\pi}{2}]$. Fix $0 < \varepsilon << 1$. For $|\beta_i| \in [0,\varepsilon]$ we have that $\{(x,y,0) : \sqrt{x^2+y^2} \le \cos\varepsilon \sin\alpha_i \cos\alpha_i A_i\} \subset \tilde{M}_i$. For $|\beta_i| \in [\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}]$ we conclude that $\{(0,0,tA_i) : |t| \in [0,\cos\varepsilon]\} \subset \tilde{M}_i$. For $|\beta_i| \in [\varepsilon, \frac{\pi}{2}-\varepsilon]$ we consider sub-cases determined by the value of α_i. If $\alpha_i \in [0,\varepsilon]$ we have $\{(x,0,z) : \sqrt{x^2+z^2} \le \frac{\cos\varepsilon |\cos\beta_i||\sin\beta_i|}{\sqrt{\cos^2\varepsilon\cos^2\beta_i + \sin^2\beta_i}} A_i\} \subset \tilde{M}_i$. If $\alpha_i \in [\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}]$, then we have that $\{(0,y,z) : \sqrt{y^2+z^2} \le \frac{\cos\varepsilon |\cos\beta_i||\sin\beta_i|}{\sqrt{\cos^2\varepsilon\cos^2\beta_i + \sin^2\beta_i}} A_i\} \subset \tilde{M}_i$. If $\alpha_i \in [\varepsilon, \frac{\pi}{2}-\varepsilon]$, then $\min_{\mathbb{S}^2}\rho_i \ge \frac{1}{\sqrt{2}}\sin^4\varepsilon \max_{\mathbb{S}^2}\rho_i$, which is a contradiction to $\min_{\mathbb{S}^2}\rho_i \le R_2$ for i large. Hence, by applying Propositions 4.1 or 4.2 we get the desired result.

Let $P = (x_P, y_P, z_P) \in \mathbb{S}^2$ be a minimum point for ρ. We define $P^1, \ldots, P^4$ as above. The tangent planes to M at $\rho(P^i)P^i$ are supporting hyperplanes for M and are given by $\{\vec{X} \in \mathbb{R}^3 : (\vec{X} - \min_{\mathbb{S}^2}\rho P^i) \cdot P^i = 0\}$ for $i = 1, \cdots, 4$. The region enclosed by these planes contains M, and thus

$$M \subset \{(x,y,z) : |z| \le \frac{\min_{\mathbb{S}^2}\rho}{|z_P|}, |x| \le \frac{\min_{\mathbb{S}^2}\rho}{|x_P|}, |y| \le \frac{\min_{\mathbb{S}^2}\rho}{|y_P|}\}. \tag{4.4}$$

Now we will establish (1.10). Suppose the contrary, then there exists R_1, R_2, and families $\{K_i\}_{i=1}^{\infty}$, $\{\rho_i\}_{i=1}^{\infty}$ so that (1.5)-(1.8) of Lemma 1.2 hold, but $\varepsilon_i := \min_{\mathbb{S}^2}\rho_i \to 0$ as $i \to \infty$, where K_i is the Gauss-Kronecker curvature of $M_i := \{\rho_i(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$.

Let $\vec{x}_i$ be such that $\rho_i(\vec{x}_i) = \varepsilon_i$. By (4.4) we see that M_i lies between two parallel hyperplanes of distance $C\varepsilon_i$ apart for some C independent of i. Hence we may apply Proposition 4.3 to get the result.

4.4. **Case where G contains an element with eigenvalues 1, $\cos\theta \pm i\sin\theta$.** Here we may assume $g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G$, $\theta \in (0,\pi)$. Let

$\{e_1, e_2, e_3\}$ be a basis for $\mathbb{R}^3$ so that g has the form $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, for $\theta \in (0, \pi)$. Since $ge_3 = e_3$ we know that there exists $\gamma \neq g$ in G. We may suppose that $\gamma \neq \operatorname{diag}\{-1, -1, -1\}$, otherwise we are done by results in [16]. Moreover, we may also suppose that γ fixes some point in $\mathbb{S}^2$. Let $\{b_1, b_2, b_3\}$ be an orthonormal basis for $\mathbb{R}^3$ such that $\gamma b_3 = b_3$. We may write $b_3 = a_1e_1 + a_2e_2 + a_3e_3$. Since $\gamma e_3 \neq e_3$ and $\gamma b_3 = b_3$ we know $a_1^2 + a_2^2 \neq 0$. By otherwise replacing e_3 with $-e_3$, and e_1, e_2 with $\frac{-a_1}{\sqrt{a_1^2+a_2^2}}e_1 + \frac{-a_2}{\sqrt{a_1^2+a_2^2}}e_2$, $\frac{a_2}{\sqrt{a_1^2+a_2^2}}e_1 - \frac{a_1}{\sqrt{a_1^2+a_2^2}}e_2$ we may suppose for some $\alpha \in (0, \frac{\pi}{2}]$ we have $b_3 = -\sin\alpha e_2 + \cos\alpha e_3$.

First assume that in the $\{b_1, b_2, b_3\}$ basis $\gamma = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$, for $\phi \in (0, \pi)$. Here we may also assume $b_1 = e_1$ and $b_2 = \cos\alpha e_2 + \sin\alpha e_3$. Therefore in the $\{e_1, e_2, e_3\}$ basis we have

$$\gamma = \begin{bmatrix} \cos\phi & \cos\alpha\sin\phi & \sin\alpha\sin\phi \\ -\cos\alpha\sin\phi & \cos^2\alpha\cos\phi + \sin^2\alpha & \cos\alpha\sin\alpha(\cos\phi - 1) \\ -\sin\alpha\sin\phi & \cos\alpha\sin\alpha(\cos\phi - 1) & \sin^2\alpha\cos\phi + \cos^2\alpha \end{bmatrix}.$$

Next we suppose in the $\{b_1, b_2, b_3\}$ basis we have $\gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. With b_1, b_2, b_3 as above, in the $\{e_1, e_2, e_3\}$ basis we have

$$\gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos 2\alpha & -\sin 2\alpha \\ 0 & -\sin 2\alpha & \cos 2\alpha \end{bmatrix}.$$

Lastly we suppose that in the $\{b_1, b_2, b_3\}$ basis $\gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Moreover, for $\alpha, |\beta| \in (0, \frac{\pi}{2}]$, we may assume that $b_1 = \cos\beta e_1 + \sin\beta\cos\alpha e_2 + \sin\beta\sin\alpha e_3$, $b_2 = -\sin\beta e_1 + \cos\beta\cos\alpha e_2 + \cos\beta\sin\alpha e_3$ and $b_3 = -\sin\alpha e_2 + \cos\alpha e_3$. Therefore in the $\{e_1, e_2, e_3\}$ basis we have

$$\gamma = \begin{bmatrix} -\cos^2\beta + \sin^2\beta & -2\cos\beta\sin\beta\cos\alpha & -2\cos\beta\sin\beta\sin\alpha \\ -2\cos\beta\sin\beta\cos\alpha & 1 - 2\sin^2\beta\cos^2\alpha & -2\sin^2\beta\cos\alpha\sin\alpha \\ -2\cos\beta\sin\beta\sin\alpha & -2\sin^2\beta\cos\alpha\sin\alpha & 1 - 2\sin^2\beta\sin^2\alpha \end{bmatrix}.$$

4.4.1. *Case where $\alpha = \frac{\pi}{2}$.* First, consider the case $g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\gamma = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$, $\theta, \phi \in (0, \pi)$ are in G. For $P = (x_P, y_P, z_P)$, we know $\min_{\mathbb{S}^2 \cap \{(x,y,z):z=z_P\}} \rho \geq \frac{\rho(P)}{2}$ and $\min_{\mathbb{S}^2 \cap \{(x,y,z):y=y_P\}} \rho \geq \frac{\rho(P)}{2}$. Fix any $P \in \mathbb{S}^2 \cap \{(x, y, z) : z \geq 0\}$, then $\rho(\sqrt{1 - z_P^2}, 0, z_P) \geq \frac{\rho(P)}{2}$, and thus $\min_{\mathbb{S}^2 \cap \{(x,y,z):y=0\}} \rho \geq \frac{\rho(P)}{4}$. For any $c \in [-1, 1]$ we have $\{(x, y, z) : y = 0\} \cap \{(x, y, z) : z = c\} \neq \phi$, and thus $\min_{\mathbb{S}^2 \cap \{(x,y,z):z=c\}} \rho \geq \frac{\rho(P)}{8}$. Hence, $\min_{\mathbb{S}^2} \rho \geq \frac{1}{8} \max_{\mathbb{S}^2} \rho$.

Next, assume $g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\theta \in (0, \pi)$, $\gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ are in G. Then for any $P = (x_P, y_P, z_P) \in \mathbb{S}^2$, we know that

$$\min_{\mathbb{S}^2 \cap \{(x,y,z):z=\pm z_P\}} \rho \geq \frac{\rho(P)}{2}.$$

This follows from the fact that $\gamma P \in \{(x, y, z) : z = -z_P\} \cap \mathbb{S}^2$. The rest of the proof follows as in the case $\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\phi \neq \frac{\pi}{2}$ is in G.

Suppose $g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\gamma = \begin{bmatrix} -\cos 2\beta & 0 & -\sin 2\beta \\ 0 & 1 & 0 \\ -\sin 2\beta & 0 & \cos 2\beta \end{bmatrix}$, $\theta \in (0, \pi)$ and $|\beta| \in (0, \frac{\pi}{2}]$ are in G. If $|\beta| = \pm\frac{\pi}{2}$, then $g\gamma = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\in G$. Hence we may assume $|\beta| < \frac{\pi}{2}$.

Let $P = (x_P, y_P, z_P) \in \mathbb{S}^2 \cap \{(x, y, z) : z \geq 0\}$ be a maximum point for ρ. Let $P^* = (\text{sign}(\beta)\sqrt{1 - z_P^2}, 0, z_P)$ and observe that $\rho(P^*) \geq \frac{1}{2}\rho(P)$, $z_P - z_{\gamma P^*} \geq |\sin 2\beta|\sqrt{1 - z_P^2}$.

We will now choose $0 < \varepsilon = \varepsilon(|\beta|) << 1$ so that for some $0 < C(|\beta|) < 1$, we have $P \in \mathbb{S}^2 \cap \{(x, y, z) : z \geq 1 - \varepsilon\}$ implies $\gamma P^* \in \mathbb{S}^2 \cap \{(x, y, z) : -C(|\beta|) \leq z \leq 1 - \varepsilon\}$. If $|\beta| \in (0, \frac{\pi}{4})$ we will simply require $\varepsilon \leq 1 - |\cos 2\beta|$.

For $|\beta| \in [\frac{\pi}{4}, \frac{\pi}{2})$, we will require that $0 < \varepsilon << 1$ satisfies $\varepsilon \leq 1 - |\cos 2\beta|$ and $\sqrt{\varepsilon}\sqrt{2-\varepsilon}|\sin 2\beta| + (1-\varepsilon)|\cos 2\beta| < 1$.

Suppose first that $P \in \mathbb{S}^2 \cap \{(x,y,z) : z \geq 1-\varepsilon\}$. If $z_{\gamma P^*} \leq 0$, then we have $\{(x,y,z) : z \in [\frac{\rho_{\max} z_{\gamma P^*}}{4}, \frac{\rho_{\max} z_P}{4}], \sqrt{x^2+y^2} \leq (\frac{\rho_{\max}}{4} - z)\frac{\sqrt{1-C^2(|\beta|)}}{1+C(|\beta|)}\} \subset \tilde{M}$. If $z_{\gamma P^*} \geq 0$, then choose $Q \in \mathbb{S}^2 \cap \{(x,y,z) : z \leq 0\}$ such that $\rho(Q) = \rho_{\max}$. Thus we get $\{(x,y,z) : z \in [\frac{\rho_{\max} z_Q}{4}, \frac{\rho_{\max} z_{\gamma P^*}}{4}], \sqrt{x^2+y^2} \leq (\frac{\rho_{\max}}{4} + z)\frac{\sqrt{\varepsilon}}{\sqrt{2-\varepsilon}}\} \subset \tilde{M}$. If $P \in \mathbb{S}^2 \cap \{(x,y,z) : z < 1-\varepsilon\}$, then choose $Q \in \mathbb{S}^2 \cap \{(x,y,z) : z \leq 0\}$ such that $\rho(Q) = \rho_{\max}$. Thus we get $\{(x,y,z) : z \in [\frac{\rho_{\max} z_Q}{2}, \frac{\rho_{\max} z_P}{2}], \sqrt{x^2+y^2} \leq (\frac{\rho_{\max}}{2} + z)\frac{\sqrt{\varepsilon}}{\sqrt{2-\varepsilon}}\} \subset \tilde{M}$. In all cases we have shown that $\min_{\mathbb{S}^2 \cap \{(x,y,z):z=0\}} \rho \geq C(G) \max_{\mathbb{S}^2} \rho$, for some $C(G) > 0$. Now, $z_{\gamma(\pm 1,0,0)} = \mp \sin 2\beta$, and thus

$$\min_{\{(x,y,z):z=\pm\sin 2\beta\}\cap\mathbb{S}^2} \rho \geq \frac{C(G)}{2}\rho_{\max}$$

and thus

$$\{(x,y,z) : |z| \leq \frac{C(G)|\sin 2\beta|\rho_{\max}}{2}, \sqrt{x^2+y^2} \leq \frac{C(G)|\cos 2\beta|\rho_{\max}}{2}\} \subset \tilde{M}.$$

If $|\beta| = \frac{\pi}{4}$ then we see that

$$\{(x,y,z) : |z| \leq \frac{C(G)}{2}\rho_{\max}, \sqrt{x^2+y^2} \leq \frac{C(G)}{2}\rho_{\max} - |z|\} \subset \tilde{M},$$

and thus $\{\vec{X} \in \mathbb{R}^3 : |\vec{X}| \leq C(G) 2^{-\frac{3}{2}}\rho_{\max}\} \subset \tilde{M}$.

4.4.2. *Proof of Lemma* 1.2 *if* $\alpha < \frac{\pi}{2}$ *and* γ *has eigenvalues* 1, $a \pm ib$, $b \neq 0$ *or* $1, -1, -1$. Let $P \in \mathbb{S}^2 \cap \{(x,y,z) : z \geq 0\}$ be a maximum point of ρ. We define $P^* := (0, \sqrt{1-z_P^2}, z_P)$, $P^\dagger := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos 2\alpha & -\sin 2\alpha \\ 0 & -\sin 2\alpha & \cos 2\alpha \end{bmatrix} P^*$.

By construction, we have $P^* \in \{(x,y,z) : z = z_P\} \cap \mathbb{S}^2$, and thus $\rho(P^*) \geq \frac{1}{2}\rho(P)$ and $\rho(P^\dagger) \geq \frac{1}{4}\rho(P)$. Moreover, $z_P - z_{P^\dagger} \geq \sin(2\alpha)\sqrt{1-z_P^2}$.

First we consider the case where $\alpha \in (0, \frac{\pi}{4})$. Fix $\varepsilon = \varepsilon(\alpha)$, small and positive so that $-\sin(2\alpha)\sqrt{\varepsilon}\sqrt{2-\varepsilon} + \cos(2\alpha)(1-\varepsilon) > 0$, and $1-\varepsilon \geq \cos(2\alpha)$. If $P \in \mathbb{S}^2 \cap \{(x,y,z) : z \geq 1-\varepsilon\}$, then we have $z_{P^\dagger} \geq -\sin(2\alpha)\sqrt{\varepsilon}\sqrt{2-\varepsilon} + \cos(2\alpha)(1-\varepsilon) > 0$. Also, $z_{P^\dagger} \leq \cos(2\alpha) \leq 1-\varepsilon$.

In this case we may also choose $Q \in \mathbb{S}^2 \cap \{(x,y,z) : z \leq 0\}$ so that $\rho(Q) = \max_{\mathbb{S}^2} \rho$. By convexity, it follows that

$$\{(x,y,z) : z \in [\frac{\max_{\mathbb{S}^2} \rho}{8} z_Q, \frac{\max_{\mathbb{S}^2} \rho}{8} z_{P^\dagger}],$$

$$\sqrt{x^2+y^2} \leq \sqrt{\frac{\varepsilon}{2-\varepsilon}}(z + \frac{\max_{\mathbb{S}^2} \rho}{8})\} \subset \tilde{M}.$$

This implies

$$\min_{\mathbb{S}^2 \cap \{z=0\}} \rho \geq \sqrt{\frac{\varepsilon}{2-\varepsilon}} \frac{\max_{\mathbb{S}^2} \rho}{8}.$$

If $P \in \{(x,y,z) : 0 \leq z \leq 1-\varepsilon\} \cap \mathbb{S}^2$, then as above, we find $Q \in \{(x,y,z) : z \leq 0\} \cap \mathbb{S}^2$ satisfying $\rho(Q) = \max_{\mathbb{S}^2} \rho$. In this case we have

$$\{(x,y,z) : z \in [\frac{\max_{\mathbb{S}^2} \rho}{2} z_Q, \frac{\max_{\mathbb{S}^2} \rho}{2} z_P],$$

$$\sqrt{x^2+y^2} \leq \sqrt{\frac{\varepsilon}{2-\varepsilon}}(z + \frac{\max_{\mathbb{S}^2} \rho}{2})\} \subset \tilde{M}.$$

This implies

$$\min_{\mathbb{S}^2 \cap \{z=0\}} \rho \geq \sqrt{\frac{\varepsilon}{2-\varepsilon}} \frac{\max_{\mathbb{S}^2} \rho}{2}.$$

Now we consider the case where $\alpha = \frac{\pi}{4}$. For $P \in \{(x,y,z) : z \geq 0\} \cap \mathbb{S}^2$ a maximum point of ρ, we have $z_{P^\dagger} = -\sqrt{1 - z_P^2} \leq 0$. Fix $0 < \varepsilon << 1$. If we have $P \in \{(x,y,z) : z \geq \varepsilon\} \cap \mathbb{S}^2$, then $z_{P^\dagger} \geq -\sqrt{1-\varepsilon^2}$. Hence, by convexity, we have $\{(x,y,z) : z \in [\frac{\rho(P)}{8} z_{P^\dagger}, \frac{\rho(P)}{8} z_P], \sqrt{x^2+y^2} \leq \frac{\varepsilon}{1+\sqrt{1-\varepsilon^2}}(\frac{\rho(P)}{8} - z)\} \subset \tilde{M}$, and thus

$$\min_{\{z=0\} \cap \mathbb{S}^2} \rho \geq \frac{1}{8} \frac{\varepsilon}{1 + \sqrt{1-\varepsilon^2}} \max_{\mathbb{S}^2} \rho.$$

If we have $P \in \{(x,y,z) : 0 \leq z \leq \varepsilon\} \cap \mathbb{S}^2$, then by convexity we have

$$\{(x,y,z) : \frac{\rho(P)}{8} z_{P^\dagger} \leq z \leq \frac{\rho(P)}{8} z_P, \sqrt{x^2+y^2} \leq \frac{\sqrt{1-\varepsilon^2}}{1+\varepsilon}(\frac{\rho(P)}{8} + z)\} \subset \tilde{M},$$

and thus

$$\min_{\{z=0\} \cap \mathbb{S}^2} \rho \geq \frac{1}{8} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \max_{S^2} \rho.$$

Lastly, we consider the case where $\alpha \in (\frac{\pi}{4}, \frac{\pi}{2})$. In this case we have $z_{P^\dagger} \leq 0$. We now fix $0 < \varepsilon << 1$ satisfying $\varepsilon \leq \frac{1}{4}(\frac{1 - |\cos(2\alpha)|}{\sqrt{2}\sin(2\alpha)})^2$.

First, we consider if $P \in \{(x,y,z) : z \geq 1-\varepsilon\} \cap \mathbb{S}^2$. In such a case we have $\sqrt{1-z_P^2} \leq \sqrt{2}\sqrt{\varepsilon}$. Thus, $z_{P^\dagger} \geq -\frac{1}{2} - \frac{|\cos(2\alpha)|}{2} > -1$. By convexity, we have

$$\{(x,y,z) : \frac{\rho(P)}{8} z_{\gamma P^*} \leq z \leq \frac{\rho(P)}{8} z_P,$$

$$\sqrt{x^2+y^2} \leq \frac{2\sqrt{1-(\frac{1+|\cos(2\alpha)|}{2})^2}}{3+|\cos(2\alpha)|}(\frac{\rho(P)}{8} - z)\} \subset \tilde{M}.$$

Hence,

$$\min_{\{z=0\}\cap\mathbb{S}^2} \rho \geq \frac{1}{8}\frac{\sqrt{1-(\frac{1+|\cos(2\alpha)|}{2})^2}}{3+|\cos(2\alpha)|}\max_{\mathbb{S}^2}\rho.$$

If $P \in \{(x,y,z) : 0 \leq z \leq 1-\varepsilon\} \cap \mathbb{S}^2$, then we have $z_{P^\dagger} \leq 0$. It follows that

$$\{(x,y,z) : \frac{\rho(P)}{8} z_{P^\dagger} \leq z \leq \frac{\rho(P)}{8} z_P, \sqrt{x^2+y^2} \leq \sqrt{\frac{\varepsilon}{2-\varepsilon}}(z + \frac{\rho(P)}{8})\} \subset \tilde{M},$$

which implies that

$$\min_{\{z=0\}\cap\mathbb{S}^2} \rho \geq \frac{1}{8}\sqrt{\frac{\varepsilon}{2-\varepsilon}}\max_{\mathbb{S}^2}\rho.$$

If γ has eigenvalues $1, a \pm ib$, $b \neq 0$, then $z_{\gamma(\pm 1,0,0)} = \mp \sin\alpha \sin\phi$. Moreover, $0 < \sin\alpha\sin\phi < 1$. Hence

$$\min_{\{z=\pm\sin\alpha\sin\phi\}\cap\mathbb{S}^2} \rho \geq \frac{1}{2}C(G)\max_{\mathbb{S}^2}\rho.$$

Convexity then yields

$$\{(x,y,z) : |z| \leq \frac{C(G)}{2}\sin\alpha\sin\phi\max_{\mathbb{S}^2}\rho,$$

$$\sqrt{x^2+y^2} \leq \frac{C(G)}{2}\sqrt{1-\sin^2\alpha\sin^2\phi}\max_{\mathbb{S}^2}\rho\} \subset \tilde{M}.$$

If γ has eigenvalues $1, -1, -1$, then we have $z_{\gamma(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0)} = \mp\frac{\sin(2\alpha)}{\sqrt{2}}$. Moreover, $0 \neq \sin(2\alpha)$. Thus, we get

$$\min_{\{z=\pm\frac{1}{\sqrt{2}}\sin(2\alpha)\}\cap\mathbb{S}^2} \rho \geq \frac{1}{2}C(G)\max_{\mathbb{S}^2}\rho.$$

Convexity yields

$$\{(x,y,z) : |z| \leq \frac{C(G)}{2\sqrt{2}}\sin(2\alpha)\max_{S^2}\rho,$$

$$\sqrt{x^2+y^2} \leq \frac{C(G)}{2}\sqrt{1-\frac{1}{2}\sin^2(2\alpha)}\max_{\mathbb{S}^2}\rho\} \subset \tilde{M}.$$

4.4.3. *Proof of Lemma* 1.2 *when* $\alpha < \frac{\pi}{2}$ *and* γ *has eigenvalues* $-1,1,1$. Let $P \in \{(x,y,z) : z \geq 0\} \cap \mathbb{S}^2$ be a maximum point of ρ. We define $P^* := (0, \sqrt{1-z_P^2}, z_P)$, and thus $\rho(P^*) \geq \frac{1}{2}\rho(P)$. Moreover, $z_P - z_{\gamma P^*} \geq \sin^2\beta\sin(2\alpha)\sqrt{1-z_P^2}$, $z_{\gamma P^*} \leq \max\{0, 1-2\sin^2\beta\sin^2\alpha\}$, and $1 - 2\sin^2\beta\sin^2\alpha > -1$, since $\alpha < \frac{\pi}{2}$.

We wish to fix $0 < \varepsilon << 1$ small so that for $P \in \{(x,y,z) : z \geq 1-\varepsilon\} \cap \mathbb{S}^2$ we have $z_{\gamma P^*} \geq -\sin^2\beta\sin^2\alpha > -1$. Thus we will require that $\varepsilon < 1 - \frac{1}{\sqrt{1+[\frac{\sin^2\beta\sin(2\alpha)}{2\sin^2\beta\sin^2\alpha-1}]^2}}$ and $1 - \sin^2\beta\sin^2\alpha + \varepsilon(2\sin^2\beta\sin^2\alpha - 1) - \sqrt{\varepsilon}\sqrt{2-\varepsilon}\sin^2\beta\sin(2\alpha) \geq 0$.

Consider the case where $1 - 2\sin^2\beta\sin^2\alpha < 0$. In such a case $z_{\gamma P^*} \leq 0$. If $P \in \{(x,y,z) : z \geq 1-\varepsilon\} \cap \mathbb{S}^2$, then we have $z_{\gamma P^*} \geq -\sin^2\alpha\sin^2\beta$. Therefore, by convexity

$$\{(x,y,z) : \frac{\rho(P)}{4}z_{\gamma P^*} \leq z \leq \frac{\rho(P)}{4}z_P,$$

$$\sqrt{x^2+y^2} \leq (\frac{\rho(P)}{4} - z)\frac{\sqrt{1-\sin^4\alpha\sin^4\beta}}{1+\sin^2\alpha\sin^2\beta}\} \subset \tilde{M},$$

and thus

$$\min_{\{z=0\}\cap S^2}\rho \geq \frac{\max_{S^2}\rho}{4}\frac{\sqrt{1-\sin^4\alpha\sin^4\beta}}{1+\sin^2\alpha\sin^2\beta}.$$

If we have $P \in \{(x,y,z) : 0 \leq z \leq 1-\varepsilon\} \cap \mathbb{S}^2$, then we have

$$\{(x,y,z) : \frac{\rho(P)}{4}z_{\gamma P^*} \leq z \leq \frac{\rho(P)}{4}z_P, \sqrt{x^2+y^2} \leq (\frac{\rho(P)}{4} + z)\sqrt{\frac{\varepsilon}{2-\varepsilon}}\} \subset \tilde{M},$$

and thus

$$\min_{\{z=0\}\cap\mathbb{S}^2}\rho \geq \frac{\max_{\mathbb{S}^2}\rho}{4}\sqrt{\frac{\varepsilon}{2-\varepsilon}}.$$

Now suppose $1 - 2\sin^2\beta\sin^2\alpha \geq 0$. Assume that $P \in \{(x,y,z) : z \geq 1-\varepsilon\} \cap \mathbb{S}^2$. If $z_{\gamma P^*} \geq 0$, we may choose $Q \in \{(x,y,z) : z \leq 0\} \cap \mathbb{S}^2$ so that $\rho(Q) = \max_{\mathbb{S}^2}\rho$. Thus, we have $\rho|_{[\{z=z_{\gamma P^*}\}\cup\{z=z_Q\}]\cap\mathbb{S}^2} \geq \frac{1}{8}\max_{\mathbb{S}^2}\rho$. Moreover, $z_{\gamma P^*} \leq 1 - 2\sin^2\beta\sin^2\alpha$. Thus, by convexity

$$\{(x,y,z) : \frac{\rho_{\max}}{8}z_Q \leq z \leq \frac{\rho_{\max}}{8}z_{\gamma P^*},$$

$$\sqrt{x^2+y^2} \leq (z + \frac{\rho_{\max}}{8})\frac{\sqrt{1-(1-2\sin^2\beta\sin^2\alpha)^2}}{2-2\sin^2\beta\sin^2\alpha}\} \subset \tilde{M},$$

and thus

$$\min_{\{z=0\}\cap\mathbb{S}^2} \rho \geq \frac{\max_{\mathbb{S}^2}\rho}{8}\frac{\sqrt{1-(1-2\sin^2\beta\sin^2\alpha)^2}}{2-2\sin^2\beta\sin^2\alpha}.$$

If $z_{\gamma P^*} < 0$, then we know $\rho|_{[\{z=z_P\}\cup\{z=z_{\gamma P^*}\}]\cap\mathbb{S}^2} \geq \frac{\max_{\mathbb{S}^2}\rho}{4}$, and thus by convexity we have

$$\{(x,y,z) : \frac{\rho_{\max}}{4}z_{\gamma P^*} \leq z \leq \frac{\rho_{\max}}{4}z_P,$$

$$\sqrt{x^2+y^2} \leq (\frac{\rho_{\max}}{4} - z)\frac{\sqrt{1-\sin^4\alpha\sin^4\beta}}{1+\sin^2\alpha\sin^2\beta}\} \subset \tilde{M},$$

and thus

$$\min_{\{z=0\}\cap\mathbb{S}^2} \rho \geq \frac{\max_{\mathbb{S}^2}\rho}{4}\frac{\sqrt{1-\sin^4\alpha\sin^4\beta}}{1+\sin^2\alpha\sin^2\beta}.$$

If $P \in \{(x,y,z) : z \leq 1-\varepsilon\}\cap\mathbb{S}^2$, then we choose $Q \in \{(x,y,z) : z \leq 0\}\cap\mathbb{S}^2$ so that $\rho(Q) = \max_{\mathbb{S}^2}\rho$. In such a case, by convexity, we have

$$\{(x,y,z) : \frac{\rho_{\max}}{2}z_Q \leq z \leq \frac{\rho_{\max}}{2}z_P, \sqrt{x^2+y^2} \leq (\frac{\rho_{\max}}{2} + z)\sqrt{\frac{\varepsilon}{2-\varepsilon}}\} \subset \tilde{M},$$

and thus

$$\min_{\{z=0\}\cap\mathbb{S}^2} \rho \geq \frac{\max_{\mathbb{S}^2}\rho}{2}\sqrt{\frac{\varepsilon}{2-\varepsilon}}.$$

We have $z_{\gamma(0,\pm1,0)} = \mp\sin^2\beta\sin(2\alpha)$. Moreover, $0 < \sin^2\beta\sin(2\alpha) < 1$ unless $\alpha = \frac{\pi}{4}$ and $|\beta| = \frac{\pi}{2}$, and hence

$$\min_{\{z=\pm\sin^2\beta\sin(2\alpha)\}\cap\mathbb{S}^2} \rho \geq \frac{C(G)}{2}\max_{\mathbb{S}^2}\rho.$$

Thus, by convexity, we have

$$\{(x,y,z) : |z| \leq \frac{C(G)}{2}\sin^2\beta\sin(2\alpha)\rho_{\max},$$

$$\sqrt{x^2+y^2} \geq \frac{C(G)}{2}\sqrt{1-\sin^4\beta\sin^2(2\alpha)}\rho_{\max}\} \subset \tilde{M}.$$

If $\alpha = \frac{\pi}{4}$ and $|\beta| = \frac{\pi}{2}$ we choose $0 < \theta < \frac{\pi}{2}$ and thus $z_{\gamma(\cos\theta,\pm\sin\theta,0)} = \mp\sin\theta$. Hence $\min_{\{z=\pm\sin\theta\}\cap\mathbb{S}^2}\rho \geq \frac{C(G)}{2}\rho_{\max}$, and by convexity

$$\{(x,y,z) : |z| \leq \frac{C(G)}{2}\sin\theta\rho_{\max}, \sqrt{x^2+y^2} \geq \frac{C(G)}{2}\cos\theta\rho_{\max}\} \subset \tilde{M}.$$

5. Higher Regularity and Existence through Degree Theories

In this section, we present the degree theory argument for the existence of a solution to (1.4), as in [16]. Let us define

$$h(\vec{x},u,\nabla u,\nabla^2 u) := (\det e_{ij})^{-1}u^{n+2}(u^2+|\nabla u|^2)^{-\frac{n}{2}-1}\det(\nabla_{ij}u+ue_{ij}), \quad (5.1)$$

where $e_{ij} =< e_i, e_j >$, $e_1, \cdots, e_n$ denote a local frame of $\mathbb{S}^n$, and ∇ represents covariant differentiation on $\mathbb{S}^n$ with respect to the standard metric. We are going to solve

$$h(\vec{x},u,\nabla u,\nabla^2 u) = F(\frac{1}{u(\vec{x})}\vec{x}), \quad \vec{x}\in\mathbb{S}^n. \quad (5.2)$$

We then define $\rho = \frac{1}{u}$ to obtain our desired hypersurface $M = \{\rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^n\}$. For $t \in [0,1]$ we define

$$F_t(\vec{X}) := tF(\vec{X}) + (1-t) \quad \vec{X}\in\mathbb{R}^{n+1}\setminus\{0\}, \quad (5.3)$$

and consider the operators

$$\mathcal{F}(\cdot,t) : C^{4,\alpha}_G(\mathbb{S}^n) \to C^{2,\alpha}_G(\mathbb{S}^n)$$

defined by

$$\mathcal{F}(u,t) := h(\vec{x},u,\nabla u,\nabla^2 u) - F_t(\frac{1}{u(\vec{x})}\vec{x}), \quad (5.4)$$

where we have $0 < \alpha < 1$ and

$$C^{4,\alpha}_G(\mathbb{S}^n) = \{u\in C^{4,\alpha}(\mathbb{S}^n) : u(g\vec{x}) = u(\vec{x}) \quad \forall g\in G\},$$
$$C^{2,\alpha}_G(\mathbb{S}^n) = \{u\in C^{2,\alpha}(\mathbb{S}^n) : u(g\vec{x}) = u(\vec{x}) \quad \forall g\in G\}.$$

5.1. **Uniqueness of solutions to $\mathcal{F}(u,0) = 0$ and invertibility of $\mathcal{F}_u(1,0)$.** Clearly $u \equiv 1$ satisfies $\mathcal{F}(u,0) = 0$. A theorem of A.D. Alexandrov asserts that compact hypersurfaces with Gauss-Kronecker curvature identically equal to 1 are unit spheres. A computation shows $\mathcal{F}_u(1,0)\varphi = \triangle\varphi + n\varphi$. The kernel of $\triangle + n$ is the span of spherical harmonics of degree one. Since G has no fixed points on $\mathbb{S}^n$, the only G-invariant element in the kernel of $\triangle + n$ is the zero element. Thus $\mathcal{F}_u(1,0)$ is invertible.

5.2. **Higher Order Estimates.** Lemma 1.2 and convexity imply that there is a C so that $\|u\|_{C^1(\mathbb{S}^n)} \leq C$. Lemma 7.2.1 of Oliker [18] implies that for some C we have $\|u\|_{C^2(\mathbb{S}^n)} \leq C$. Lemma 17.14 of [7], or equivalently results of Evans [6] and Krylov [13], implies that $\|u\|_{C^{2,\alpha}(\mathbb{S}^n)} \leq C$, for some $0 < \alpha < 1$ and positive C. Lemma 17.16 of [7] then implies $\|u\|_{C^{4,\alpha}(\mathbb{S}^n)} \leq C$, for some C and $0 < \alpha < 1$.

5.3. **Existence Results via Degree Theory.** Degree theory for second-order fully nonlinear elliptic operators on Riemannian manifolds was given in [15]. The degree is homotopy invariant among second-order fully nonlinear elliptic operators. It follows from [15], with slight modifications, that we can define a degree theory for group invariant second-order fully nonlinear elliptic operators on Riemannian manifolds. This degree is also homotopy invariant among group invariant second-order fully nonlinear elliptic operators. Let C be the constant so that $\frac{1}{C} < u < C$ and $\|u\|_{C^{4,\alpha}(\mathbb{S}^n)} < C$, and set

$$\mathcal{O} := \{u \in C_G^{4,\alpha}(\mathbb{S}^n) : \frac{1}{C} < u < C, \|u\|_{C^{4,\alpha}(\mathbb{S}^n)} < C\}.$$

Thus, we have

$$\deg(\mathcal{F}(\cdot, 1), \mathcal{O}, 0) = \deg(\mathcal{F}(\cdot, 0), \mathcal{O}, 0).$$

Using that $u \equiv 1$ is the unique positive solution of $\mathcal{F}(u, 0) = 0$ in $C_G^{4,\alpha}(\mathbb{S}^n)$, and $\mathcal{F}_u(1, 0)$ is an invertible operator from $C_G^{4,\alpha}(\mathbb{S}^n) \to C_G^{2,\alpha}(\mathbb{S}^n)$, we may adapt the methods in [15] to the group invariant situation to obtain

$$\deg(\mathcal{F}(\cdot, 0), \mathcal{O}, 0) = (-1)^i,$$

where i is the number of negative eigenvalues of $\triangle + n$ in $C_G^{4,\alpha}(\mathbb{S}^n)$. Hence

$$\deg(\mathcal{F}(\cdot, 1), \mathcal{O}, 0) \neq 0.$$

Hence $\mathcal{F}(u, 1) = 0$ has at least one solution in $\mathcal{O}$.

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