Math 102: Problem Solving

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Mathematics and Problem Solving

Four Step Problem-Solving Process*

1. Understand the Problem.

   (a) State the problem in your own words.

   (b) What are you trying to find or do?

   (c) What are the unknowns?

   (d) What information do you obtain from the problem?

   (e) What information, if any, is missing or not needed?

*Similar to the mathematician Polya's problem-solving process.

(f) Do not impose conditions that do not exist.

(g) Strip the problem of irrelevant details.

(h) Prepare a visual representation.

2. Devise a Plan.

(a) Prepare a visual representation.

(b) Eliminate impossible situations.

(c) Look for a pattern and use inductive reasoning.

(d) Examine related problems, and see if the same technique applied to them is applicable to the current problem.

(e) Examine a simpler or a special case of the problem to gain insight.

(f) Restate the problem.

(g) Guess a solution (if possible), check it, and revise your guess.

(h) Work the problem backwards.

(i) Design a model of the problem.

(j) Use an algorithm.

3. Carry out the Plan.

   (a) Implement strategies in step 2 and perform any necessary actions or computations.

   (b) Check each step of the plan.

   (c) Keep a record of your work.

4. Look Back.

   (a) Check results in original problem.

   (b) Interpret the solution in terms of the original problem. Does the answer make sense or is it reasonable? Does it answer the question that was asked?

   (c) Determine whether there is another method of finding the solution.

   (d) Determine other related or more general problems for which the techniques used will work.

Example: The natural numbers $\mathbb{N}$ are the numbers

$$1, 2, 3, 4, 5, 6, 7, \cdots$$

Find the sum of the first 100 natural numbers.

Understanding the problem: Here we must compute

$$1 + 2 + 3 + 4 + \cdots + 99 + 100$$

Devising a plan: We will use a trick which the mathematician Carl Gauss is credited with discovering as a child. We simple write out twice the sum. Let us also call the sum $s = 1 + 2 + \cdots + 100$. Then we can obtain $2s$ as follows:

$$s = 1 + 2 + 3 + 4 + \cdots + 99 + 100$$

$$s = 100 + 99 + 98 + 97 + \cdots + 2 + 1$$

$$- -$$

$$2s = \underbrace{101 + 101 + 101 + 101 + \cdots + 101 + 101}_{100 \quad terms}$$

Carrying out the plan: Thus we see that

$$2s = \underbrace{101 + 101 + \cdots + 101 + 101}_{100 \quad terms}$$

$$= 100 \cdot 101.$$

Hence

$$s = \frac{100 \cdot 101}{2} = 5050.$$

Looking back: We can use this to solve more general problems such as

$$1 + 2 + 3 + 4 + \cdots + n - 1 + n.$$

Example: Cookies are sold in packages of two or 6. How many ways can you buy a dozen cookies?

Understanding the problem: We need to find how to write

$$12 = 2n + 6m$$

where n, m are non-negative integers.

Devising a plan: Fix one of the free variables, say n, and determine the value(s) of the other variable m, if any, will work. Using $n = 0, 1, 2, 3, \cdots$

Carrying out the plan:

- $n = 0$:

  $$12 = 2 \cdot 0 + 6m = 6m$$

  implies $m = 2$.

- $n = 1$:

  $$12 = 2 \cdot 1 + 6m$$

  implies

  $$10 = 6m$$

  or

  $$m = \frac{10}{6}$$

  which is not an integer.

- $n = 2$:

  $$12 = 2 \cdot 2 + 6m$$

  implies

  $$8 = 6m$$

or

$$m = \frac{8}{6}$$

which is not an integer.

- $n = 3$:

$$12 = 2 \cdot 3 + 6m$$

implies

$$6 = 6m$$

or

$$m = \frac{6}{6} = 1.$$

- $n = 4$:

$$12 = 2 \cdot 4 + 6m$$

implies

$$4 = 6m$$

or

$$m = \frac{4}{6}$$

which is not an integer.

- $n = 5$:

$$12 = 2 \cdot 5 + 6m$$

implies

$$2 = 6m$$

or

$$m = \frac{2}{6}$$

which is not an integer.

- $n = 6$:

$$12 = 2 \cdot 6 + 6m$$

implies

$$0 = 6m$$

or

$$m = 0.$$

Looking back: Perhaps it is easier to fix m and determine n.

Explorations with Patterns:

Inductive reasoning is the method of making generalizations based on observations and patterns.

Inductive reasoning may lead to new discoveries. However, its weakness is that conclusions are drawn from only the collected evidence. Thus, if not all the cases have been checked, there is the possibility that another case will prove the conclusion false.

Inductive reasoning may lead to a **conjecture,** which this is a statement that is thought to be true, but not yet proven to be true or false.

When we find an example that contradicts the conjecture, we provide a **counterexample** and prove the conjecture false in general.

Some Homework Exercises:

1. Show how 11 can be expressed as the sum of two consecutive integers. Do you think it is possible to write any odd number as the sum as two consecutive integers? State a conjecture, and prove or disprove it.

2. On a balance scale, two spools and one thimble balances eight buttons. Also, one spool balances one thimble and one button. How many buttons will balance one spool? **Answer:** 3

3. If possible, find an odd number that can be expressed as the sum of four consecutive integers. If this is not possible, please explain why.

4. Given 3 X's and 3 O's arranged as follows

$$XXXOOO$$

By switching two adjacent letters at a time, rearrange to the following:

$$XOXOXO.$$

What is the minimum number of moves needed to do this? Suppose that you start with

$$XXXXOOOO$$

and want to end up with

$$XOXOXOXO,$$

how many moves does it take? Can you generalize your method of counting steps to the case where there are n X's and O's. Try this with $n = 5, 6, 7$, etc. **Answer:** 3, 6

5. Using a 3 minute and 5 minute hourglass, how can you measure a one minute time interval (accurately)?

6. Find the sums (indirectly of course)

$$2 + 4 + 6 + 8 + \cdots + 100$$

$$1 + 3 + 5 + 7 + \cdots + 99$$

$$7 + 8 + 9 + \cdots + 99 + 100$$

Answers: 2550, 2500, 5029

Sequences:

A **sequence** is an ordered arrangement of numbers, figures, or objects. A sequence has terms identified as first, second, third, etc.

For example:

- 1,2,3,4,5,6,$\cdots$
- A,B,C,D,E,$\cdots$
- 1,3,5,7,9,$\cdots$

are all sequences.

An **arithmetic sequence** is one in which a term is obtained from the previous term by either adding or subtracting a fixed quantity.

For example

$$1, 2, 3, 4, 5, 6, \cdots,$$

$$1000, 990, 980, 970, \cdots,$$

and

$$1, 3, 5, 7, 9, \cdots$$

are arithmetic sequences.

A **recursively defined sequence** is a sequence in which the first several terms are defined, and the remaining terms are determined from earlier terms. For example

$$1, 2, 3, 5, 8, 13, 21, \cdots$$

is such a sequence. We can define the first two terms as $1, 2$ and subsequent terms are obtained by taking the sum of the previous two terms in the sequence.

A **geometric sequence** is one where the ratio of two consecutive terms in the sequence is a specific number. For example

$$2, 6, 18, 54, \cdots$$

is a geometric sequence. Here the ratio of one term and its predecessor is 3. Another example of a geometric sequence is

$$1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots.$$

Some Homework Exercises:

1. A tank contains 15360 gallons of water. At the end of each day, one half of the water is removed. After 10 days how much water is remaining? **Answer:** 15 gallons

2. Consider the sequences

   $$300, 500, 700, 900, \cdots$$

   $$2, 4, 8, 16, \cdots$$

   How many terms does it take for the arithmetic sequence to be smaller than the geometric sequence? **Answer:** 12

3. Find the next three terms in the sequence

   $$2, 3, 6, 11, 18, 27, 38, \cdots$$

   Answer: 51, 66, 83

Algebraic Thinking:

On the set of real numbers $\mathbb{R}$, there are two very important binary operations, namely $+$ which is called **addition** and $\cdot$ (or $\times$) which is called **multiplication.** Paired with addition is the operation $-$ which is called **subtraction**, and paired with multiplication is the operation $\div$ which is called **division.**

The subject of **Algebra** is the study of the set of real numbers along with these binary operations. Let us now discuss the important properties of theses operations, and how they are related.

Because the binary operations $+$ and $\cdot$ are binary operations, they take two inputs and return one output. Thus, if you wish to add or multiply more than two real numbers, you need to be sure that this is well-defined (that is, makes sense or is possible). This is what the property of **associativity** insures:

$$a + (b + c) = (a + b) + c$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for any three real numbers a, b, c. Here we note as well that parentheses are used in mathematics to emphasize operations which must be computed first.

Note that subtraction and division are not associative. For example,

$$2 - (3 - 1) = 2 - 2 = 0,$$

whereas

$$(2 - 3) - 1 = -1 - 1 = -2.$$

Also,

$$6 \div (4 \div 2) = 6 \div 2 = 3,$$

whereas

$$(6 \div 4) \div 2 = \frac{3}{2} \div 2 = \frac{3}{4}.$$

There are two special real numbers, namely 0 and 1 in the sense that for any real number a we have

$$a + 0 = a,$$

$$a \cdot 1 = a.$$

The relationship between $+$ and $-$ is given by

$$a + (-b) = a - b$$

for any two real numbers a, b, where here $-b$ is the negative of b in the left hand side of the equality. Likewise, the relationship between $\cdot$ and $\div$ is given by

$$a \cdot \frac{1}{b} = \frac{a}{b}$$

where $a, b \neq 0$ are real numbers, and $\frac{1}{b}$ is the multiplicative reciprocal of b.

Observe as well that

$$a - a = 0$$

for any real number a and

$$a \cdot \frac{1}{a} = 1$$

for any $a \neq 0$.

Let us now discuss how the binary operations $+$ and $\cdot$ are related. The relation that relates these operations is the so-called **distributive property**, which states

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c),$$

where a, b, c are any real numbers.

Finally, to end our brief introduction to the subject of Algebra on the set of real numbers $\mathbb{R}$, we will talk about notation. Since the operations $+, -, \cdot, \div$ are binary operations, that is they take two inputs, and return an output, we must develop a consistent notation to describe how to perform successive computations using many inputs and various operations.

One such notational convention is the use of parentheses, as discussed above. More generally, we have a convention, which is called the **Order of Operations***.

*Remember: Please Excuse My Dear Aunt Sally!

The Order of Operations is as follows:

1. Operations in parentheses are performed first, working with the inner most first. In the innermost parentheses, or in the absence of parentheses, follow the remaining steps.

2. Powers are computed first, going from left to right (square roots and such are considered powers).

3. Multiplication and division are performed next (with equal weight) going from left to right.

4. Addition and subtraction are performed after all multiplications and divisions have been performed. Addition and subtraction (which hold equal weight) are performed from left to right.

Example:

$$2 + 7 \cdot 10 \div (2 + 8) = 2 + 7 \cdot 10 \div (10)$$

$$= 2 + 70 \div 10$$

$$= 2 + 7$$

$$= 9.$$

Some Important Algebraic Properties of the Real Numbers:

- *The Addition Property and Cancellation Property of Equality:* For* any real numbers a, b, c

$$a = b \iff a + c = b + c.$$

- *The Multiplication Property and Cancellation Property of Equality:* For any numbers a, b and $c \neq 0$†

$$a = b \iff a \cdot c = b \cdot c.$$

*$\Rightarrow$ reads implies and $\iff$ reads if and only if

†If you don't assume $c \neq 0$, then you may only conclude the $\Rightarrow$ implication.

Example: Pluto, once thought to be the planet furthest from the sun in our solar system, was also thought to be the smallest planet in our solar system. The table below gives the weight P of an object on Pluto and the weight E of the same object on Earth. We shall assume the units of weight (which are units of force) are the same for both measurements.

E	1	3	10	100
P	0.04	0.12	0.4	4

Find a formula for P in terms of E.

Solution:

We shall look for a linear relationship. That is, one of the form

$$P = m \cdot E + b.$$

Note that if the relationship is not linear, we will see it when we test our formula by using the values of E and P from our data table.

Using that two distant points uniquely determine a line, we simply must use two of the points (E, P) from our data table to find m and b.

Using $E = 1$ $P = 0.04$ we get

$$0.04 = m \cdot 1 + b.$$

This allows us to write $b = 0.04 - m$, and thus in general

$$P = mE + 0.04 - m.$$

Next, using $E = 3, P = 0.12$ we get

$$0.12 = m(3) + 0.04 - m,$$

and thus

$$0.08 = 2m.$$

This yields

$$m = 0.04.$$

Thus, we obtain $b = 0$, and in general we have

$$P = 0.04E.$$

Now, we test the other values in out table. If $E = 10$, our formula $P = 0.04E$ requires $P = 0.04 \cdot 10 = 0.4$, which agrees with our original table. If $E = 100$, our formula yields $P = 0.04 \cdot 100 = 4$, which also agrees with our original table.

Some Homework Exercises:

1. Peter and Paul have a total of 400 dollars. Peter has 3 times as much money as Paul. How much do they each have? **Answer:** 100, 300

2. A man has an estate worth 64,000 dollars. The eldest child received three times as much as the youngest, and the middle child received 14,000 more than the youngest. How much did each child receive? **Answer:** 10,000; 24,000; 30,000