Math 102: Sets, Functions and Reasoning

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Sets:

A **set** is a collection of objects, and the objects in the set are often called its **elements**. The symbol $\in$ is often used to show that an object x is an element of a set S. That is,

$$x \in S$$

reads x *is an element of* S.

There are several ways one describe a given set; one way is to list the set explicitly. It is common to surround the list of elements in the set by braces {, }.

For example,

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

is set which consists of the digits we use in our number system.

However, if a set does not have a finite number of elements in it, we often list the elements in a sequence to exhibit the pattern, and then use $\cdots$ to show that the pattern continues. For instance

$$\{1, 2, 3, 4, 5, \cdots\}$$

is the set of **natural numbers** or counting numbers. Often the symbol $\mathbb{N}$ is used to represent this set.

This notation can be used if the pattern continues in more than one direction, for instance

$$\{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$$

represents the set of **integers**. Often the symbol $\mathbb{Z}$ is used to represent the set of integers.

Another common way to describe a set is to use the so-called **set-builder** notation to describe it. For example:

$$\{\frac{a}{b} : a, b \text{ are integers with } b \neq 0\}$$

is the set of **rational numbers**, which is also denoted by $\mathbb{Q}$. This is usually read as *the set of fractions $\frac{a}{b}$ such that a, b are integers with $b \neq 0$*. In particular the : is usually read as *such that*. The symbol $|$ may also be used instead of : to mean such that.

Recall that it is common to use the symbol $\in$ to represent *is an element of* or *is in*.

Using $\mathbb{Z}$ to represent the set of integers

$$\{0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \cdots\}$$

which can also be represented as

$$\{n : n = 0, \text{ or } n \in \mathbb{N}, \text{ or } -n \in \mathbb{N}\}.$$

Thus we can also describe $\mathbb{Q}$ as

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}.$$

The set which contains no elements is called the **empty set**. This set is denoted by either

$$\{\}$$

or

$$\emptyset.$$

Here we must be careful that our description is precise enough. For instance, the collection

$$\{x : x \quad is \quad a \quad chair\}$$

is not considered a set since everyone may not be in agreement of what a chair is. For instance, is a stool a chair?

Thus, in set theory it is important that our collection of objects to be well-defined in order to be considered a set. If there is any ambiguity in our description of the collection, we shall not consider it a set.

Building New Sets From Existing Sets:

A set A is said to be a **subset** of a set B, which is denoted by

$$A \subseteq B,$$

if for every $x \in A$ we have

$$x \in A \quad implies \quad x \in B.$$

That is every element of A is also an element of B.

For any set A we have

$$\emptyset \subseteq A$$

and

$$A \subseteq A.$$

A set A is said to be a **proper subset** of a set B, which is denoted by

$$A \subset B,$$

if $A \subseteq B$, but there is an element $b \in B$ with $b \notin A$ (here $\notin$ reads "is not in" or "is not an element of").

Two sets are said to be **equal** if they have the same elements.

That is

$$A = B \quad \iff \quad A \subseteq B \quad and \quad B \subseteq A.$$

For example,

$$\{1, 3, 5, 7\}$$

and

$$\{n \in \mathbb{N} : n \text{ is odd and } n < 9\}$$

are equal sets.

Russell's Paradox and the Universe of Discourse:

Russell's Paradox deals with the following: Can the collection of all sets be a set itself?

We suppose that the set of all sets is a set.* We define an *ordinary set* to be a set which is not an element of itself. Consider

$$S = \{x : x \quad is \quad an \quad ordinary \quad set\}.$$

Is $S \in S$? Or is $S \notin S$? If $S \in S$, then S is not ordinary, and this cannot happen. Thus $S \notin S$, but then S is an ordinary set. Thus, we have obtained a contradiction since neither possibility can happen.

*We are actually using a so-called **proof by contraction**.

Thus, the collection of all sets can not be a set itself.

This tells us that in set theory, we must always define a set of reference, our **universe** or **universal set.** We then view every set in the particular discussion as a subset of this set U. Moreover, it also tells us that there is no universal set that will work for all situations we study in set theory.

Given a set A, we define the the **complement** of A, denoted by

$$A'$$

to be the set

$$\{x \in U : x \notin A\}.$$

Given two sets A and B, we may define the **union** of the two sets, denoted by

$$A \cup B,$$

to be the set

$$\{x \in U : x \in A \quad or \quad x \in B\}.$$

Similarly, we can define the **intersection** of A and B, denoted by

$$A \cap B,$$

to be the set

$$\{x \in U : x \in A \quad and \quad x \in B\}.$$

Given two sets A, B we may define the **set difference** or **complement of a set relative to another set** as

$$A \setminus B = \{x \in U : x \in A \quad and \quad x \notin B\}.$$

Thus, we see that

$$U \setminus A = A'.$$

Often, we use **Venn Diagrams** to visualize the relation between two sets A, B in a fixed universal set U.

Here we represent the universal set as a rectangle, and all sets A are thought of as regions in this rectangle (often drawn as circles).

We shall discuss this further in lecture.

The following are usually referred to as **De-Morgan's laws** for set-theory:

$$(A \cup B)' = A' \cap B',$$

and

$$(A \cap B)' = A' \cup B'.$$

There is a so-called **distributive property** which relates unions and intersections:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Two sets S and R are said to be in **one-to one correspondence** if there is a pairing of elements between the two sets so that each element of S is paired with one element of R and vice versa.

For example:

$$S = \{1, 2, 3\}, \quad R = \{a, b, c\}$$

are in one-to one correspondence. To see this, simply take $1 \leftrightarrow a$, $2 \leftrightarrow b$ and $3 \leftrightarrow c$ to see such a pairing. *

*Here $\leftrightarrow$ indicates the pairing.

Also

$$\mathbb{N} = \{1, 2, 3, 4, 5, \cdots\}$$

and

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \cdots\}$$

can be placed in one-to-one correspondence. You may do this by

$$1 \leftrightarrow 0, 2 \leftrightarrow 1, 3 \leftrightarrow -1$$

$$4 \leftrightarrow 2, 5 \leftrightarrow -2, 6 \leftrightarrow 3$$

$$7 \leftrightarrow -3, 8 \leftrightarrow 4, 9 \leftrightarrow -4, \cdots$$

Two sets A and B are said to be **equivalent** if they can be be put in one-to-one correspondence with one another. We commonly denote A *is equivalent to* B by

$$A \sim B.$$

Our earlier examples of R and S are examples of similar sets. That is

$$\{1, 2, 3\} \sim \{a, b, c\}$$

and

$$\mathbb{N} \sim \mathbb{Z}.$$

Cardinality:

Two sets A and B that are equivalent are said to have the same **cardinality**.

A set A is said to have **cardinality** m, with $m \in \mathbb{N}$, if

$$A \sim \{1, 2, 3, 4, 5, 6, 7, \cdots, m\}.$$

Often, we denote the cardinality of A by $n(A)$. We also define the cardinality of the empty set, $n(\emptyset)$, to be 0.

A set S is said to be **finite** if it has a finite number of elements in it. That is S is finite if

$$n(S) \in \{0, 1, 2, 3, 4, 5, 6, 7, \cdots\}.$$

Note that two equivalent finite sets have the same number of elements, namely that of their cardinality.

A set is said to be **infinite** if it is not finite.*

*An infinite set can be characterized by the property: A is infinite if and only if there is a proper subset $B \subset A$ with $B \sim A$. For example the set of even natural numbers is a proper subset of $\mathbb{N}$. Moreover the set of even naturals is equivalent to $\mathbb{N}$.

Given two finite sets A and B, there is a connection between $n(A)$, $n(B)$, $n(A \cap B)$ and $n(A \cup B)$ which is given by:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

To see this, one may use a Venn diagram as a visual aide.

Two sets A and B are said to be **disjoint** if

$$A \cap B = \emptyset.$$

In such a case $n(A \cap B) = 0$.

Some Homework Exercises:

1. If $n(A) = 10$ and $n(B) = 12$ how large can $n(A \cap B)$ be? How small can $n(A \cup B)$ be? How large can $n(A \cup B)$ be? Suppose you knew that $n(A \cap B) = 3$, what is $n(A \cup B)$?
   Answers: 10; 12; 22; 19

2. Let

   $$A = \{1, 3, 5, 7, 9\}, \quad B = \{0, 2, 4, 6, 7, 8, 9\}$$

   Find $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$
   Answers: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $\{7, 9\}$, $\{1, 3, 5\}$, $\{0, 2, 4, 6, 8\}$.

3. In the above exercise (number 2), suppose

   $$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Find $A', B', A' \cap B', A' \cup B'$. **Answer:** $\{0, 2, 4, 6, 8\}$; $\{1, 3, 5\}$; $\emptyset$; $\{0, 1, 2, 3, 4, 5, 6, 8\}$

4. Describe a one to one correspondence between $\mathbb{N}$ and $\mathbb{Z}$.

5. Describe a one to one correspondence between $\mathbb{N}$ and $\{n \in \mathbb{N} : n \quad is \quad even\}$.

Functions and Graphs:

A **function** is a rule that assigns to each element of a set A a unique element of another set B. The set A is called the **domain** of the function, and the collection of elements of B that are paired with elements in A are the **range** of the function. The set B is often called the **codomain** of the function. We note that the range is always a subset of the codomain.

The **value of a function** is the element in the range for a particular element in the domain. *

*If $x \in domain$, then $f(x)$ represents the value of the function f paired with x.

Often a letter is used to denote a given function, for instance f. Thus, the notation

$$f : A \to B$$

reads f *is a function from* A *to* B. That is f is a function whose domain is A, and whose codomain is B.

Example: Let A be the collection of students at Lock Haven University who have completed at least one semester, and B the set of real numbers. We shall define some functions $f : A \to B$.

- Assign to each student the student's g.p.a.

- Assign to each student the student's student I.D. number.

- Assign to each student the student's number of credit hours completed.

When our function f is a function whose domain and range are sets of numbers, it is often possible to have a formula for the value of the function in terms of the element x in the domain. The notation $f(x)$ represents this value.

Here let A and B be collection of real numbers, and consider the following examples of functions

- $f(x) = x$

- $f(x) = 3x + 1$

- $f(x) = x^2$

- $f(x) = x^3$

The **graph** of a function f whose domain is A and whose range is a subset of B is the collection

$$\{(x, f(x)) : x \in A\}.$$

We shall draw the graphs of several functions in the lecture.

Some Homework Exercises:

1. Suppose that

$$(1,1),(2,3),(4,7),(5,9)$$

are the elements of a graph of a function f. Find the domain and range of f, and plot these points. Can you find a formula for $f(x)$? **Answer:** $\{1,2,4,5\}$; $\{1,3,7,9\}$; $f(x) = 2x - 1$

2. A repair person charges 25 dollars for each repair under 1 hour, and 15 for each hour after the first hour spent on the repair. Find a formula for the repair bill in terms of the number of hours x spent of a repair (assume $x > 1$ of course). If a repair bill was 30 dollars, how much time was spent on the repair? **Answer:** $f(x) = 25 + 15(x - 1) = 15x + 10$; 1 hour and 20 minutes.

Mathematical Logic:

A **statement** or **proposition** is a sentence that is either true or false.

For example

$$3 + 2 = 5$$

and

$$2 + 2 = 5$$

are both statements, however

$$x + 2 = 5$$

is not a statement.

From a given statement, one can always form a new statement by forming the **negation.** A **denial** of a proposition is a form of the negation of the proposition.

For example, for the two statements

$$2 + 3 = 5, \quad 2 + 2 = 5$$

the negation of these can be written as

$$2 + 3 \neq 5, \quad 2 + 2 \neq 5$$

respectively.

For instance, for the statement:

All of Dr. Mikula's math 102 students love math 102.

The negation can be written as:

There is a student of Dr. Mikula, taking math 102, that does not love math 102.

The above example is an example of a statement with **quantifiers.**

Some examples of quantifiers are: *all*, *some*, *every*, *for all*, *there exists*, *there doesn't exist*, *none are*, *no*, *none*, *some are*, etc.

These quantifiers are applied to elements in some set of reference or **universe of discourse**.

- **All, every, none, no** refer to every element in your reference set or universe of discourse, and thus they are commonly referred to as **universal quantifiers.**

- **Some, there exists, not all** refer to one or more elements of your reference set or universe of discourse, and thus they are commonly referred to as **existential quantifiers.**

- **All, every, each** have the same mathematical meaning; **some, there exists at least one** also have the same mathematical meaning.

Truth Tables:

If P represents a statement or proposition, we will use the notation $\sim P$ to represent the negation of P. We shall use F to represent False, and T to represent True for truth values of statements or propositions. The following **Truth Table** summarizes the relationship between the truth values for the propositions P and $\sim P$.

P	$\sim P$
T	F
F	T

Compound Statements or Propositions:

Given two propositions P, Q, we can form the so-called **conjunction** of P and Q, denoted by

$$P \wedge Q,$$

which is the proposition P and Q. Similarly, we can form the so-called **disjunction** of P and Q, denoted by

$$P \vee Q,$$

which is the proposition P or* Q.

*this is the mathematical inclusive or

The following truth tables summarize the connection between the truth values of P and Q and the truth values of $P \wedge Q$ and $P \vee Q$.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Equivalence of Propositions:

Two statements or propositions are said to be **logically equivalent** if they have the same truth tables. For instance, the propositions $(\sim P)\wedge(\sim Q)$ and $\sim(P\vee Q)$ are logically equivalent, since

P	Q	$\sim P$	$\sim Q$	$(\sim P)\wedge(\sim Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

P	Q	$P\vee Q$	$\sim(P\vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

.

The above example of logical equivalence of

$$\sim (P \vee Q) \quad \text{and} \quad (\sim P) \wedge (\sim Q)$$

is often called **DeMorgan's Law,** along with the local equivalence of

$$\sim (P \wedge Q) \quad \text{and} \quad (\sim P) \vee (\sim Q).$$

We note that $\equiv$ shall mean *are logically equivalent*.

Conditionals and Biconditionals:

Given two propositions P, Q, the proposition "if P, then Q", denoted by

$$P \Rightarrow Q$$

is called a **conditional** proposition. The proposition P is called the **hypothesis** or **antecedent** of the conditional, and the proposition Q is called the **conclusion** or **consequent** of the conditional.

Given two propositions P, Q, the proposition "P if and only if Q", denoted by

$$P \iff Q$$

is called a **biconditional** proposition.

The following truth tables show the truth values for the conditional and biconditional propositions given above:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note that

$$P \iff Q \quad \equiv \quad (P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

Use

$$P \Rightarrow Q$$

to translate:

If P, then Q.
P implies Q.
P is sufficient for Q.
P only if Q.
Q, if P.
Q whenever P.
Q is necessary for P.
Q, when P.

Use

$$P \iff Q$$

to translate:

P if and only if Q.
P if, but only if, Q.
P is equivalent to Q.
P is necessary and sufficient for Q.

The Order of Operations We Follow in Symbolic Logic:

- You examine the propositions in parentheses first, working from the inside out. In the innermost parentheses, or in the absence of parentheses, follow the remaining steps.

- $\sim$ is performed first, going from left to right.

- $\vee$ and $\wedge$ are performed second (with equal weight), going from left to right.

- $\Rightarrow$ is performed third, going from left to right.

- $\iff$ is performed last, going from left to right. *

*When in doubt, use parentheses!

Given a conditional

$$P \Rightarrow Q,$$

the **converse** is

$$Q \Rightarrow P,$$

the **contrapositive** is

$$\sim Q \Rightarrow \sim P,$$

and the **inverse** is

$$\sim P \Rightarrow \sim Q.$$

The conditional and its contrapositive are logically equivalent.

The inverse and the converse are logically equivalent.

The conditional and its converse are not logically equivalent.

Another logical equivalence to observe that relates a conditional proposition to a disjunction is

$$P \Rightarrow Q \quad \equiv \quad (\sim P) \vee Q.$$

Valid Reasoning:

In problem solving, the reasoning is said to be **valid** if the conclusion necessarily follows from the hypothesis. In lecture we shall discuss so-called **Euler diagrams** and consider examples of direct reasoning, which may use Euler diagrams as an aide in determining if the argument posed is valid.

One such example is the following argument:

Hypothesis: All roses are red.

This flower is a rose.

Conclusion: Therefore, this flower is red.

Notice that in the above example, the hypothesis statement *All roses are red,* is not actually a true statement. However, an argument is valid, if assuming the hypothesis **is true**, we may conclude the conclusion.

The so-called **law of detachment** or **modus ponens** is a direct reasoning argument that has the form:

Hypothesis: If P, then Q.

P

Conclusion: Therefore, Q.

The formal justification of this following from the truth table for the proposition

$$[(P \Rightarrow Q) \wedge P] \Rightarrow Q$$

which is given below:

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge P$	$[(P \Rightarrow Q) \wedge P] \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The fact that the truth table contains all T's in the rightmost column says that the proposition

$$[(P \Rightarrow Q) \wedge P] \Rightarrow Q$$

is a **tautology.**

The so-called **modus tollens** is an example of an indirect reasoning argument, and has the form:

Hypothesis: If P, then Q.

$\sim Q$

Conclusion: Therefore, $\sim P$.

The formal justification of this following from the truth table for the proposition

$$[(P \Rightarrow Q) \wedge \sim Q] \Rightarrow \sim P$$

which is given below:

P	Q	$P \Rightarrow Q$	$[(P \Rightarrow Q) \wedge \sim Q] \Rightarrow \sim P$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

The fact that the truth table contains all T's in the rightmost column says that the proposition

$$[(P \Rightarrow Q) \wedge \sim Q] \Rightarrow \sim P$$

is a **tautology.**

Some Homework Exercises:

1. Create a truth table for $\sim P \wedge Q$. Then use DeMorgan's law to find an equivalent proposition. **Answer:** The columns are P: TTFF, Q:TFTF and the final column is FFTF; $\sim (P \vee \sim Q)$

2. Is the following argument valid? Please justify:

   All P's are Q's; Some Q's are R's. Therefore, Some P's are R's.
   Answer: No

3. Deduce a valid conclusion:

If it is raining, then Tom carries his umbrella.
Tom is not carrying his umbrella.
Answer: It is not raining.

4. In the above exercise, can you conclude that if Tom is carrying his umbrella that it must be raining? **Answer:** No, this is the so-called fallacy of the converse.

5. Find the inverse, converse and contrapositive of the following conditional statement:

   If a shirt is red, then it is a t-shirt.

   Answers: If a shirt is not red, then it is not a t-shirt. If a shirt is a t-shirt, then it is red. If a shirt is not a t-shirt, then it is not red.