Math 102: Numeration Systems and Whole-Number Computations

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Numeration Systems:

There are several ways that cultures, present and past, use(d) characters or symbols to represent numbers. Essentially, there is a finite collection of characters or symbols, and there is a convention that allows one to use the characters or symbols to write other numbers that there aren't a specific symbol for.

Such a collection of characters or symbols, along with the convention of using these characters or symbols to write other numbers, is usually revered to as a **numeration system.**

Most ancient systems used characters whose meanings and values were always the same.

In **additive notation**, characters represent specific numbers, and if listed from from left to right, (in either increasing or decreasing order), the values of each character used are summed to get the overall value.

For instance if $i = 1$, $t = 10$ and $h = 100$ we may write

$$hhtttiiiiii$$

to represent

236.

To avoid number presentations of inordinate length, the additive method can be changed into a **subtractive method.** Such a system places smaller values before larger values to represent subtraction, when the larger values are typically written on the left, the smaller value is written to the left of it.

For instance

$$hthiit$$

represents

198.

In **multiplicative notation** two kinds of operations are employed, one kind being constant and its effective value being increased, being multiplied by symbols of another kind, also of constant meaning and value. An example of such a system is the Chinese system.

A **positional or place-value system** is another kind. We shall discuss this system in much detail, and the system in use today in the U.S. is such a system.

The **Hindu-Arabic System of Numeration** is the **numeral system** that is in use today in many parts of the world, including here in the U.S.

The Hindu-Arabic system uses ten **digits**

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

and these digits correspond with cardinalities of the following sets:

Numbers for Digits	Cardinality of the following sets
0	$\emptyset$ or $\{\}$
1	$\{a\}$
2	$\{a, b\}$
3	$\{a, b, c\}$
4	$\{a, b, c, d\}$
5	$\{a, b, c, d, e\}$
6	$\{a, b, c, d, e, f\}$
7	$\{a, b, c, d, e, f, g\}$
8	$\{a, b, c, d, e, f, g, h\}$
9	$\{a, b, c, d, e, f, g, h, i\}$

The written symbols

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

are called **numerals**.

Different cultures developed different systems to represent cardinal numbers.

The Hindu-Arabic systems, using **numerals** to represent cardinal numbers greater than nine, requires a **numeration system.**

A **numeration system** is a collection of properties and symbols agreed upon to represent numbers systematically.

An example of an additive numeration system is the **Egyptian Numeration System**. This system developed some 5400 years ago, and is based on a tally system with grouping properties. The grouping of ten symbols for one character, which represents a specific natural number, such as one, is then given a new symbol. For instance, if | represents one, then we have:

Number	**Symbol**
1	$\mid$
2	$\mid\mid$
3	$\mid\mid\mid$
4	$\mid\mid\mid\mid$
5	$\mid\mid\mid\mid\mid$
6	$\mid\mid\mid\mid\mid\mid$
7	$\mid\mid\mid\mid\mid\mid\mid$
8	$\mid\mid\mid\mid\mid\mid\mid\mid$
9	$\mid\mid\mid\mid\mid\mid\mid\mid\mid$
10	$\cap$
20	$\cap\cap$
21	$\cap \cap \mid$
90	$\cap \cap \cap \cap \cap \cap \cap \cap \cap$
100	*9*

Hence, the Egyptian use a so-called **base-ten** numeration system.

Another numeration system that developed around the time of the Egyptian Numeration System is the **Babylonian Numeration System.** However, the Babylonian system of numeration was a base-sixty system, which used a **place value system.**

The **Roman Numeration System** is an example of a subtractive system. The following represents some symbols for cardinal numbers in the Roman system of numeration, along with the Hindu-Arabic equivalents.

Hindu-Arabic Numeral	**Roman Numeral**
1	I
5	V
10	X
50	L
100	C
500	D
1000	M

For instance, in the Roman system of numeration

$$1984 = MCMLXXXIV.$$

In the Roman system, the symbols V, L, D may not be used for subtraction. Only one symbol of the symbols I, X, C may be placed before a higher valued symbol.

For instance

Number	Roman Numeral
4	IV
9	IX
40	XL
90	XC
99	XCIX
900	CM

In the middle ages the subtractive Roman Numeral System was adapted to create a multiplicative system. For instance $\bar{M}$ is used to represent $1000 \cdot 1000$ or $1000^2 = 1,000,000$. Also $\bar{V}$ is used to represent $5 \cdot 1000 = 5000$ and $\overline{CDX}$ is used to represent $410 \cdot 1000 = 410,000$.

The Hindu-Arabic Numeral system was developed by Hindus and was transported to Europe by the Arabs. This system is based on powers of ten, and so it is called a **base-ten** or **decimal** system.

Before we begin to discuss this system, we need to use an exponential notation to represent powers of a number a, where $a \neq 0$. We define $a^0 = 1$, $a^1 = a$, and $a^{n+1} = a^n \cdot a$ for any $n \in \mathbb{N}$. Thus we have

$$a^n = \underbrace{a \cdot a \cdots a}_{n \quad times}.$$

If we use t to represent ten, where ten is the cardinal number of the set

$$\{a, b, c, d, e, f, g, h, i, j\},$$

then we write cardinal numbers bigger than nine as a sum of digits times powers of ten, where here $1 = t^0$ is also considered a power of ten.

For instance, the number t or ten itself is

$$1 \cdot t^1 + 0$$

and is symbolized by 10. The number two hundred and seventy five is

$$2 \cdot t^2 + 7 \cdot t + 5$$

and is written as 275.

In the Hindu-Arabic system, the **place value** assigns a value to a digit based on its position in a numeral. To find the value of a digit in a numeral, we simply multiply the **face value** – that is, the digit's value – with the place value of the the digit in a numeral.

For instance, in the number

$$1984,$$

the place value of 4 is 1, the place value of 8 is 10, the place value of 9 is 100 and the place value of 1 is 1000.

The face value of 4 is 4, the face value of 8 is 8, the face value of 9 is 9 and the face value of 1 is 1

The value of 4 is $4 \cdot 1 = 4$, the value of 8 is $8 \cdot 10 = 80$, the value 9 is $9 \cdot 100 = 900$ and the value of 1 is $1 \cdot 1000 = 1000$. By summing the values of the digits, we get

$$1000 + 900 + 80 + 4 = 1984.$$

By writing the $1000, 100, 10$ as powers of 10, we see

$$1984 = 1 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10^1 + 4 \cdot 1.$$

This agrees with how to write 1984 by the method discussed above, if we write t in place of 10.

Some Homework Exercises:

1. Find the Egyptian and Roman Numerals for

$$648 \quad and \quad 649.$$

Answers: $\mathit{999999} \cap\cap\cap\cap||||||||$, $DCXLVIII$; $\mathit{999999} \cap\cap\cap\cap|||||||||$, $DCXLIX$;

2. Find the Roman Numerals for 2009 and 3,345,669.
Answers:$MMIX$; $\overline{MMMCCCXLV}DCLXIX$.

3. What Hindu-Arabic Numeral does

$$5t^4 + 3t^2 + 7$$

represent? (here t is ten). **Answer:** 50,307

Extra Credit Assignment:

The following assignment can add 5 pts to your overall grade, if it is done to my satisfaction.

The assignment is to write a paper, with total length being at least 3 pages, and should include a formal works cited page. In the paper discuss at least one numeration system developed from one culture in each column in the following table. In your paper you should discuss where and when these systems were/are in use:

Hindu and Arabic	Egyptian
Chinese	Greek
Roman	Mayan
Hebrew	Sumerian or Babylonian

If you choose to write about the Hindu-Arabic system you must discuss the Arabic system as used today in the middle east, as well as its historical roots in India. You must discuss what system is currently in use in India today. Is this system the same as the Arabic system? What are their numerals? What was the original system that was used in India that developed into any of the present day systems? Also, you must discuss when this system began being used in Europe, in place of earlier systems such as the Roman system.

If you discuss the Egyptian or Roman systems please discuss their conventions to deal with numbers other than the natural numbers, such as rational numbers (in general). For example, is there a way to write $\frac{1}{2}$ or $\frac{2}{5}$?

The Euclidean or Division Algorithm:

Given two natural numbers a, b with $a \leq b$, it is possible to find a unique

$$r \in \{0, 1, 2, \cdots, a-1\}$$

and a unique positive integer k so that

$$b = k \cdot a + r.$$

For example, given 21, 78 we may write

$$78 = 3 \cdot 21 + 15.$$

The Binary System of Numeration:

The binary system of numeration is a positional system that is base-two. Thus, the digits used in the binary system are $0, 1$, which are the possible remainders when diving by 2. This system is used by computers to record data and perform calculations because a switch representing on or off can be represented by the numbers 1 or 0 respectively. If we were to use T to represent two, then the number

$$1001110_{two}$$

would mean

$$1 \cdot T^6 + 0 \cdot T^5 + 0 \cdot T^4 + 1 \cdot T^3 + 1 \cdot T^2 + 1 \cdot T + 0 \cdot 1$$

$$= 64 + 8 + 4 + 2$$

$$= 78_{ten}.$$

Going from Base-ten to Binary:

Suppose we wanted to convert a base-ten number to a binary number equivalent, how would we do it?

To begin, we first consider powers of 2 in base-ten, as shown in the table below:

Power of 2	Decimal Equivalent
2^0	1
2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64
2^7	128
2^8	256
2^9	512
2^{10}	1024

We begin by showing an example:

Suppose you want to convert 125 (125_{ten}) to binary.

First you apply the division algorithm to 125 and the largest power of 2 less than 125. This gives you

$$125 = 1 \cdot 64 + 61.$$

Then we take the remainder, 61 and the largest power of 2 less than it, in this case 32, and apply the division algorithm to get

$$61 = 1 \cdot 32 + 29.$$

Continuing, we get

$$29 = 1 \cdot 16 + 13,$$

$$13 = 1 \cdot 8 + 5,$$

$$5 = 1 \cdot 4 + 1.$$

Therefore,

$$125_{ten} = 1 \cdot 64 + 1 \cdot 32 + 1 \cdot 16 + 1 \cdot 8 + 1 \cdot 4 + 1$$

$$= 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$$

$$= 1111101_{two}.$$

Going between Base-Five and Base-Ten:

The digits in base-five are the possible remainders when you divide a whole numbers by 5, and they are

$$\{0, 1, 2, 3, 4\}.$$

We shall now discuss how to go between base-five and base-ten.

Suppose you have a number in base-five Hindu-Arabic positional notation, and you wish to convert it to base-ten Hindu-Arabic positional notation. How may this be accomplished? Let us consider an example so to illustrate the general procedure.

$$
\begin{aligned}
34101_{five} &= 3 \cdot 5^4 + 4 \cdot 5^3 + 1 \cdot 5^2 + 0 \cdot 5^1 + 1 \cdot 5^0 \\
&= 3 \cdot 625 + 4 \cdot 125 + 25 + 1 \\
&= 1875 + 500 + 25 + 1 = 2401_{ten}.
\end{aligned}
$$

To convert a number from base-ten to base-five, we need to use the Euclidean Algorithm. Before we start, we need a table of powers of 5 to refer to:

Power of 5	Decimal Equivalent
5^0	1
5^1	5
5^2	25
5^3	125
5^4	625
5^5	3125
5^6	15625

Consider 7890_{ten}, we shall now use the Euclidean Algorithm to convert this to base-five notation. We begin by first selecting the largest power of 5 less than or equal to 7890, and in this case this is $5^5 = 3125$. Thus

$$7890 = 2 \cdot 5^5 + 1640.$$

Next, we take the remainder and apply the Euclidean Algorithm to that with the largest power of 5 less than or equal to it, namely $5^4 = 625$. Thus,

$$1640 = 2 \cdot 5^4 + 390.$$

Continuing in this manner, we get

$$390 = 3 \cdot 5^3 + 15,$$

$$15 = 3 \cdot 5^1.$$

Thus,

$$7890_{ten} = 2 \cdot 5^5 + 2 \cdot 5^4 + 3 \cdot 5^3 + 3 \cdot 5^1 + 0 \cdot 5^0$$

$$= 223030_{five}.$$

Going between Base-Twelve and Base-Ten:

The digits in base-twelve are the possible remainders when you divide a whole numbers by 12, and they are

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E\},$$

where $T = 10_{ten}$ and $E = 11_{ten}$.

We shall now discuss how to go from base-twelve to base-ten.

Suppose you have a number in base-twelve Hindu-Arabic positional notation, and you wish to convert it to base-ten Hindu-Arabic positional notation. How may this be accomplished? Let us consider an example so to illustrate the general procedure.

$$E0T4_{twelve} = 11 \cdot 12^3 + 0 \cdot 12^2 + 10 \cdot 12 + 4$$

$$= 11 \cdot 1728 + 0 + 120 + 4$$

$$= 19132_{ten}.$$

To convert a number from base-ten to base-twelve, we need to use the Euclidean Algorithm. Before we start, we need a table of powers of 12 to refer to:

Power of 12	Decimal Equivalent
12^0	1
12^1	12
12^2	144
12^3	1728
12^4	20736

Consider $12,456_{ten}$, we shall now use the Euclidean Algorithm to convert this to base-twelve notation. We begin by first selecting the largest power of 12 less than or equal to 12456, and in this case this is $12^3 = 1728$.

Thus,

$$12,456 = 7 \cdot 12^3 + 360.$$

Next, we choose the largest power of 12 less than 360, in this case 144. Again by the Euclidean algorithm, we get

$$360 = 2 \cdot 12^2 + 72$$

and

$$72 = 6 \cdot 12.$$

Thus,

$$12,456_{ten} = 7 \cdot 12^3 + 2 \cdot 12^2 + 6 \cdot 12$$

$$= 7260_{twelve}.$$

Some Homework Exercises:

1. Convert 1236_{ten} to binary and base-five notations. **Answers:** 10011010100_{two}; 14021_{five}

2. Find the decimal equivalents of 1443_{five} and 1001110_{two}. **Answers:** 248_{ten}; 78_{ten}

3. Find the decimal equivalent of $E97_{twelve}$ and find 20000_{ten} in base-twelve notation. **Answers:** 1699_{ten}; $E6T8_{twelve}$

Addition and Subtraction of Whole Numbers:

Earlier we defined the set of **Natural Numbers** to be the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \cdots\}.$$

If we add on zero, we get the set $\mathbb{N} \cup \{0\}$, which is called the set of **Whole Numbers**. That is, the set of Whole Numbers is the set

$$\mathbb{W} = \{0, 1, 2, 3, \cdots\}.$$

We may think of the set of Whole Numbers as the set of possible cardinality values for finite sets.

Given this set $\mathbb{W}$, we may define a binary operation* on $\mathbb{W}$ called **addition**. We shall do so as follows:

*Recall that a binary operation is an operation that has two inputs.

Let

$$a, b \in \mathbb{W}$$

we may define a set A with $n(A) = a$, and a set B, with A and B disjoint, satisfying $n(B) = b$. We then define the addition of a and b, denoted by

$$a + b,$$

by

$$a + b = n(A \cup B).$$

The numbers a and b are usually referred to as the **addends**, and the quantity $a + b$ is usually referred to as the **sum of** a **and** b. Moreover, the binary operation of taking sums is usually referred to as **addition.**

Given two whole numbers a, b we have what is called the **closure property of addition.** What is says is that

$$a + b$$

is a unique whole number. Moreover, we have **commutativity of addition,** which says that

$$a + b = b + a.$$

Given 3 whole numbers a, b, c, we may define

$$a + (b + c)$$

to be the unique whole number that you obtain by first computing $(b + c)$ and then adding a and $b + c$. Likewise, we may define

$$(a + b) + c$$

to be the unique whole number you obtain by first computing $a + b$ and the adding $a + b$ and c. The so-called **associative property of addition** says

$$a + (b + c) = (a + b) + c$$

and this allows us to define summation for more that two addends, without ambiguity.

Another property of $\mathbb{W}$ with the operation $+$ is the the so-called **identity property of addition**. This says that there is a unique whole number 0, the **additive identity**, which has the property

$$a + 0 = a = 0 + a$$

for all $a \in \mathbb{W}$.

The Take-Away Model for Subtraction:

Operations that undo one another are often called **inverse operations**. *Subtraction may be thought of as the inverse operation of addition.*

You may model the operation of subtraction using cardinality of sets as follows: For $m, n \in \mathbb{W}$ with m no larger than n, you simply take a set A with $n(A) = n$ and a subset $B \subseteq A$ with $n(B) = m$, then

$$n - m = n(A \setminus B) = n(A \cap B').$$

The Missing-Addend Model for Subtraction:

The **missing-addend** model for subtraction relates addition and subtraction. If you want to subtract one whole number m from a larger whole number n you first call the supposed difference x. That is,

$$x = n - m.$$

Then you rewrite this as

$$x + m = n.$$

The Comparison Model for Subtraction:

Another model from subtraction is the so-called comparison model. This model compares two collections or sets with different numbers of elements.

For instance if $m, n \in \mathbb{W}$ with $n \geq m$ we may find a collection A with n elements in it and a collection B with m elements. If we were to find a one-to-one correspondence between B and some proper subset of A, how many things in A would not be paired with an element of B?

Of course the answer is $n - m$.

The Number Line Model for Addition and Subtraction:

This method will be discussed in lecture.

Algorithms for Addition and Subtraction of Whole Numbers:

We shall use *base-ten blocks* to illustrate whole number addition, and thus based on this illustration, discuss the **Right-to-Left** and **Left-to-Right Algorithms for Addition.**

For instance, if we let B represent a base-ten *block* * which has 1000 units, F represents a *flat* which has 100 units, L represents a *long* of 10 units and U represents a single unit†, then we can add by noting that

$$10U = L, \quad 10L = F, \quad 10F = B.$$

*this is also called a long-flat

†This will be done with visuals or manipulative in the class room.

For instance, if you wanted to add

$$568 + 757,$$

you first express each number as the appropriate number of $U's, L's, F's$ and $B's$. That is

$$568 = 5F + 6L + 8U$$

and

$$757 = 7F + 5L + 7U.$$

Adding these together, you get

$$568 + 757$$

$$= (5F + 6L + 8U) + (7F + 5L + 7U)$$

$$= (5 + 7)F + (6 + 5)L + (8 + 7)U$$

Now $(8+7)U = 15U = L+5U$, since $10U = L$, and thus

$$(5 + 7)F + (6 + 5)L + (8 + 7)U$$

$$= (5 + 7)F + (6 + 5 + 1)L + 5U.$$

Also, $(6 + 5 + 1)L = 12L = F + 2L$, and thus

$$(5 + 7)F + (6 + 5 + 1)L + 5U$$

$$= (5 + 7 + 1)F + 2L + 5U.$$

Finally, $(5 + 7 + 1)F = (13)F = B + 3F$ and thus

$$(5 + 7 + 1)F + 2L + 5U$$

$$= B + 3F + 2L + 5U = 1325.$$

Hence

$$568 + 757 = 1325.$$

We shall also discuss this example using the Left-to-Right Algorithm. This is done below:

$$\begin{array}{cccc} & 5 & 6 & 8 \\ + & 7 & 5 & 7 \\ \hline 1 & 2 & & \\ & 1 & 1 & \\ & & 1 & 5 \\ \hline 1 & 3 & 2 & 5 \end{array}$$

Let us now do another example using this algorithm:

	1	9	8	4
+		6	6	6
	1			
	1	5		
		1	4	
			1	0
	2	6	5	0

Consider another example:

$$
\begin{array}{ccccc}
 & 1 & 2 & 5 & 9 \\
+ & 2 & 7 & 4 & 3 \\
\hline
 & 3 & & & \\
 & & 9 & & \\
 & & & 9 & \\
 & & & 1 & 2 \\
\hline
\end{array}
$$

What do we do next?

We continue to add down in each column, going from left to right, to get:

$$
\begin{array}{ccccc}
 & 1 & 2 & 5 & 9 \\
+ & 2 & 7 & 4 & 3 \\
\hline
 & 3 & & & \\
 & & 9 & & \\
 & & & 9 & \\
 & & & 1 & 2 \\
\hline
 & 3 & & & \\
 & & 9 & & \\
 & & 1 & 0 & \\
 & & & & 2 \\
\hline
\end{array}
$$

We then continue to get:

$$
\begin{array}{ccccc}
 & 1 & 2 & 5 & 9 \\
+ & 2 & 7 & 4 & 3 \\
\hline
 & 3 & & & \\
 & & 9 & & \\
 & & & 9 & \\
 & & & 1 & 2 \\
\hline
 & 3 & & & \\
 & & 9 & & \\
 & & 1 & 0 & \\
 & & & & 2 \\
\hline
 & 3 & & & \\
 & 1 & 0 & & \\
 & & & 0 & \\
 & & & & 2 \\
\hline
 & 4 & 0 & 0 & 2
\end{array}
$$

Because this involves many steps, we may use the more traditional method of summing from right to left:

$$
\begin{array}{ccccc}
 & 1 & 2 & 5 & 9 \\
+ & 2 & 7 & 4 & 3 \\
\hline
\end{array}
$$

Adding the numbers in the right-most column gives a sum of 12, we write the two below the right-most column, and ”carry” the 1 to above the column to the left of it.

$$
\begin{array}{ccccc}
 & & & {\scriptstyle 1} & \\
 & 1 & 2 & 5 & 9 \\
+ & 2 & 7 & 4 & 3 \\
\hline
 & & & & 2
\end{array}
$$

We then add the numbers in the second column in from the right to get a sum of 10, and we write a 0 below this column and carry the 1 to get:

$$\begin{array}{ccccc} & & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 1 & 2 & 5 & 9 \\ + & 2 & 7 & 4 & 3 \\ \hline & & & 0 & 2 \end{array}$$

Continuing, we get:

$$\begin{array}{ccccc} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 1 & 2 & 5 & 9 \\ + & 2 & 7 & 4 & 3 \\ \hline & & 0 & 0 & 2 \end{array}$$

And thus:

$$\begin{array}{rcccc} & ^{1} & ^{1} & ^{1} & \\ & 1 & 2 & 5 & 9 \\ + & 2 & 7 & 4 & 3 \\ \hline & 4 & 0 & 0 & 2 \end{array}$$

Algorithms for Subtraction of Whole-Numbers:

Suppose we wanted to compute

$$153 - 42,$$

how shall we proceed?

Using our positional based notation, we may interpret this as

$$(F + 5L + 3U) - (4L + 2U)$$

$$= F + 5L + 3U - 4L - 2U$$

$$= F + (5 - 4)L + (3 - 2)U$$

$$= F + L + U = 111.$$

Thus $152 - 42 = 111$.

However, suppose that we wanted to compute:

$$153 - 74.$$

Rewriting this, we get:

$$F + 5L + 3U - (7L + 4U)$$

$$= F + 5L + 3U - 7L - 4U.$$

Notice that we want to remove 7 L's and 4 U's, and we have 5 L's and 3 U's, how can this be done?

Recall that $F = 10L$, we may use this and rewrite the above as:

$$10L + 5L + 3U - 7L - 4U$$

$$= 15L + 3U - 7L - 4U$$

$$= 8L + 3U - 4U.$$

Moreover, $L = 10U$ and thus we rewrite $8L$ as $7L + 10U$ and thus

$$8L + 3U - 4U$$

$$= 7L + 13U - 4U$$

$$= 7L + 9U = 79.$$

Thus $153 - 74 = 79$.

The above example is illustrated in the usual way be perform subtraction*:

$$\begin{array}{rrrr} & 1 & 5 & 3 \\ - & & 7 & 4 \\ \hline \end{array}$$

$$\begin{array}{rrrr} & 1 & {}^{4}\searrow 5 & {}^{1}3 \\ - & & 7 & 4 \\ \hline & & & 9 \end{array}$$

$$\begin{array}{rrrr} & {}^{0}\searrow 1 & {}^{14}\searrow 5 & {}^{1}3 \\ - & & 7 & 4 \\ \hline & & 7 & 9 \end{array}$$

*Here ${}^{m}\searrow n$ means we cross out n and replace it by m.

Some Homework Exercises:

1. Perform the following additions using the left-to-right and the right-to-left (traditional) algorithms. Also, perform this addition using base ten blocks/manipulatives as an aid.

   $$347 + 485$$

   Answer: 832

2. Perform the addition using the traditional algorithm.

   $$1478 + 2392$$

   Answer: 3870

3. Perform the subtraction using ase ten manipulatives, and the usual algorithm.

   $$472 - 387$$

Answer: 85

Multiplication and Division of Whole Numbers:

In this section we shall discuss various models of multiplication and division of whole numbers.

As subtraction is the inverse operation of addition, we will see that division is the inverse operation of multiplication.

We shall also discuss the so-called **distributive property** that relates the binary operations of addition and multiplication.

We may think of multiplication of two whole numbers n, m as a repeated sum. That is

$$n \cdot m = \underbrace{m + m + m + m + \cdots + m}_{n \quad \text{times}}.$$

This way of defining multiplication, in terms of repeated addition, is called the **Repeated-Addition Model.**

We shall use three notations to represent multiplication of two numbers n, m, which we shall use interchangeably, namely

$$nm, \quad n \cdot m, \quad n \times m.$$

It turns out that

$$m \cdot n = \underbrace{n + n + n + n + \cdots + n}_{m \text{ times}}$$

is actually the same as $n \cdot m$. That is

$$n \cdot m = m \cdot n,$$

and this is the so-called **commutative property of multiplication.**

A second model for multiplication of two whole numbers is the **Array and Area Models**. This we shall discuss further in lecture, since it requires additional visual aids. However, essentially the idea is that we take a rectangle that is n units long by m units wide, and define $n \cdot m$ to be the area of this rectangle.

We may relate this property to the repeated addition model by viewing our rectangle as a bunch of squares, each of whose side's length is 1. Then the area on the rectangle is simply the sum of the areas of the squares, which is the number of squares times 1, where 1 is the area of one of the squares.

Properties of Multiplication of Whole Numbers:

- **Closure Property**: For $n, m \in \mathbb{W}$ we have
$$n \cdot m \in \mathbb{W}.$$

- **Commutative Property**: For $n, m \in \mathbb{W}$ we have
$$n \cdot m = m \cdot n.$$

- **Associativity Property**: For $n, m, k \in \mathbb{W}$ we have
$$n \cdot (m \cdot k) = (n \cdot m) \cdot k.$$

- **Identity Property**: There is a unique number $1 \in \mathbb{W}$ so that for any $n \in \mathbb{W}$ we have
$$1 \cdot n = n \cdot 1 = n.$$

- **Multiplication by Zero Property**: For any $n \in \mathbb{W}$ we have

$$0 \cdot n = n \cdot 0 = 0.$$

- **Distributive Property of Multiplication over Addition**: This property says that for $n, m, k \in \mathbb{W}$ we have

$$k \cdot (m + n) = (k \cdot m) + (k \cdot n).$$

Order of Operations:

Order of operations refers to the order which the binary operations $+$, $-$, $\cdot$, $\div$ are applied in a compound calculation or computation. This is a necessary convention, if we wish to include multiple binary computations in one single line or statement, and wish to minimize the use of parentheses.

The convention is as follows:

1. Operations in parentheses must be performed first, working from the inner-most set of parentheses out. In the inner most set of parentheses, refer to the order of operations listed in steps 2,and 3.

2. Multiplications and divisions are performed first going from left to right.*

3. Additions and subtractions are performed after all multiplications and divisions are done. Additions and subtractions are computed going from left to right.

*Usually the operation of exponentiation is done before multiplication and division. However, if the exponent is an integer, we may treat it as a multiplication or division. Thus, until we discuss non-integer exponents, this is not an issue.

Multiplication Algorithms:

We now will discuss multiplication algorithms. We shall use both $\cdot$ and $\times$ to represent the binary operation of multiplication. Let us start with an example:

$$5 \cdot 13$$

Using our manipulatives, we can think of 13 as $L + 3U$. We want to take 5 copies of these, that is

$$(L+3U)+(L+3U)+(L+3U)+(L+3U)+(L+3U)$$

$$= 5L + (3 + 3 + 3 + 3 + 3)U$$

$$= 5L + 15U$$

$$= 5L + L + 5U$$

$$= 6L + 5U = 65.$$

Hence $5 \cdot 13 = 65$ or $5 \times 13 = 65$ Symbolically this may be expressed as

$$\begin{array}{rcc} & \overset{1}{1} & 3 \\ \times & & 5 \\ \hline & 6 & 5 \end{array}$$

which shall be referred to as multiplication using the traditional algorithm.

Another way we can think of this multiplication is through the use of the distributive property, since

$$5 \cdot 13 = 5 \cdot (10 + 3)$$

$$= 5 \cdot 10 + 5 \cdot 3$$

$$= 50 + 15$$

$$= 65.$$

Let us consider 2 digit numbers mulpiplications. For example

$$22 \cdot 33$$

$$= (20 + 2) \cdot (30 + 3)$$

$$^{*} = (20 + 2) \cdot 30 + (20 + 2) \cdot 3$$

$$^{\dagger} = 20 \cdot 30 + 2 \cdot 30 + 20 \cdot 3 + 2 \cdot 3$$

$$= 600 + 60 + 60 + 6 = 726.$$

*by the distributive property

†by the distributive property

Thus $22 \cdot 33 = 726$. Symbolically, this can be expressed as:

$$\begin{array}{ccc} & 3 & 3 \\ \times & 2 & 2 \\ \hline {}_{1} & 6 & 6 \\ 6 & 6 & \\ \hline 7 & 2 & 6 \end{array}$$

Multiplication by Powers of 10:

Here we will use manipulatives, noting

$$10U = L, \quad 10L = F, \quad 10F = B.$$

Suppose for instance we want to muliply 32 by 10. If we think of 32 as

$$32 = 3L + 2U$$

we want to add up ten coplies of this. Hence, we are taking 10 copies of every maniplulative in

$$3L + 2U.$$

Thus, the $2U$ becomes $20U = 2L$ and the $3L$ becomes $30L = 3F$, and therefore

$$10 \cdot (3L + 2U) = 3F + 2L = 320.$$

We also may see this by

$$10 \cdot 32 = 10 \cdot (3 \cdot 10 + 2)$$

$$= 3 \cdot 10^2 + 2 \cdot 10 = 320.$$

We now use this to simplify muliplications of larger numbers, for example:

$$\begin{aligned} & 7 \cdot 365 \\ & = 7 \cdot (3 \cdot 10^2 + 6 \cdot 10 + 5) \\ & = 7 \cdot 3 \cdot 10^2 + 7 \cdot 6 \cdot 10 + 7 \cdot 5 \\ & = 21 \cdot 10^2 + 42 \cdot 10 + 35 \\ & = 2100 + 420 + 35 \\ & = 2555. \end{aligned}$$

Example:

$$13 \cdot 17 = 13 \cdot (10 + 7)$$

$$13 \cdot 10 + 13 \cdot 7$$

$$= 130 + (10 + 3) \cdot 7$$

$$= 130 + 70 + 21$$

$$= 221.$$

Some Homework Exercises:

1. Perform the following multiplation using base ten manipulatives and the traditional algorithm.

$$4 \times 243$$

   Answer: 972

2. Use the distributive property to evaluate

$$19 \cdot 21, \quad 18 \cdot 73$$

   and the perform these multiplications using the traditional algorithm. **Answers:** 399; 1314

Division of Whole Numbers:

For any whole numbers a, b, with $b \neq 0$

$$a \div b = c$$

if and only if there is a unique whole number c such that

$$a = b \cdot c.$$

The number a is the **dividend**, b is the **divisor**, and c is the **quotient** in $a \div b = c$.

We note that

$$a \div b$$

is sometimes written as

$$\frac{a}{b}.$$

The Set Partition Model for Division of Whole numbers:

Let a, b be whole numbers, with $b \neq 0$. We would like to model $a \div b$ using set-theoretic ideas. The way we do this is follows:

Let A be a set with cardinality $n(A) = a$. We wish to partition A into b disjoint subsets, each of which has the same cardinality, namely c. If this can be done, then $a \div b$ is defined and is equal to c.

The Missing-Factor Model for Division of Whole Numbers:

This model for division is the model we gave as our initial definition of $a \div b$. If essentially asks if there is a $c \in \mathbb{W}$ so that

$$a = b \cdot c.$$

In such a case we define

$$a \div b = c.$$

The Repeated-Subtraction Model for Division of Whole Numbers:

Suppose we have a set A with cardinality $n(A) = a$. Suppose we wish to repeated remove subsets from A, each of whose cardinality is $b \neq 0$, a whole number, until there is b elements left in A, then we may define $a \div b = c$.

That is, if we can do find disjoint sets

$$B_1, B_2, \cdots, B_c \subseteq A$$

for some $c \in \mathbb{W}$, with

$$n(B_1) = n(B_2) = \cdots n(B_c) = b$$

and

$$(\cdots((A \setminus B_1) \setminus B_2) \cdots \setminus B_{c-1}) = B_c,$$

then

$$a \div b = c.$$

The Division Algorithm:

Just as subtraction of whole numbers doesn't always result in another whole number*, the operation of division of whole numbers doesn't always result in another whole number. That is given any two whole numbers a, b

$$a - b \in \mathbb{W} \iff a \geq b$$

and

$$a \div b \in \mathbb{W} \iff a = b \cdot c$$

for some $c \in \mathbb{W}$ where for $a \div b$ we also need $b \neq 0$ (why?).

*You might end up with a negative integer.

We would like to extend the operation of division to whole numbers a, $b \neq 0$ with simply the requirement that $a \geq b$.

Even though the whole numbers are not closed under division, the operation of division is meaningful with whole numbers, even if $\frac{a}{b} = a \div b$ is not a whole number.

For example, suppose you have a collection of 33 apples, and you want to give apples to six different people in a way that you give each person the same number of apples, and as many apples as you can, how many apples should each of the six people get?

If you give each of the 6 people 5 apples, then you will give out a total of 30 apples, and there are 3 apples remaining. You cannot give each of the 6 people another apple, because you do not have enough apples remaining.

Here the number 3 is the so-called **remainder** when you divide 33 by 6.

In general, if $a \geq b$ are whole numbers, we may find a natural number q and a number $r \in \{0, 1, 2, \cdots, b-1\}$ so that

$$a = q \cdot b + r.$$

Here r is called the **remainder**, and this process of writing a in terms of b as above is called the **division algorithm** or the **Euclidean algorithm**.

Here we may use the notation

$$a \div b = q \quad R \quad r$$

to represent

$$a = q \cdot b + r.$$

This ties in to the mixed number notation for an improper fraction

$$\frac{a}{b} = q\frac{r}{b}.$$

Division by 0 and by 1:

Recall that $a \div b$ for whole numbers a and b means that there is a whole number c so that

$$a \div b = c$$

or equivalently

$$\frac{a}{b} = c.$$

Moreover, we can do this if and only if

$$a = c \cdot b.$$

Thus, if $b = 1$ we see that for any $a \in \mathbb{W}$

$$a = a \cdot 1.$$

Thus 1 divides any whole number a, and

$$a \div 1 = a.$$

Note that for any whole number a, $a \cdot 0 = 0$. Hence if 0 divides a, then $a \div 0 = c$ for some whole number c, and hence

$$a = c \cdot 0 = 0.$$

Thus, if $a \neq 0$ this is not true. Moreover $0 \div 0$ is also undefined because $a \div a$ is 1 if it is defined. Thus if $0 \div 0$ exists, it needs to be 1, and this is equivalent to

$$0 = 1 \cdot 0.$$

However, we could make $0 \div 0$ equal to any whole number c since

$$0 = c \cdot 0.$$

Thus $0 \div 0$ or $\frac{0}{0}$ is an **indeterminate** and thus it is **undefined**.

Thus

$$a \div 0$$

is not defined for any whole number a.

Algorithms for Division of Whole Numbers:

By far, the long division algorithm is the most complicated procedure in the elementary mathematics curriculum. The main idea behind the long division algorithm is the use of the Euclidean algorithm.

To gain insight into the long division algorithm, we will use base-ten blocks (manipulatives) and a fundamental definition of division.

Suppose we wish to calculate

$$461 \div 3.$$

We will first examine how this can be done using manipulatives, and this will help us understand the steps in the so-called **long division algorithm.**

Thinking of 461 as $4F + 6L + 1U$, we wish to partition a collection with 461 blocks into 3 piles, with as many blocks as possible in each pile, and the same number of blocks in each pile. In the process of splitting our 461 blocks into 3 piles, we will stop once we have split up the original collection of 461 blocks into 3 equal piles and have either 0, 1 or 2 blocks remaining.

Let us first start with the $4F$ portion of $4F + 6L + 1U$. Clearly each of out three piles can get $1F$ and no more $F's$. This follows since

$$4 = 1 \cdot 3 + 1.$$

Symbolically this is represented by

$$\begin{array}{ll} & \ \ 1 \\ 3 &)\overline{4}\ \ \ \overline{6}\ \ \ \overline{1} \\ & \ \ \underline{3} \\ & \ \ 1 \end{array}$$

Now we have $F+6L+U$ remaining; we cannot partition F further. Thus we replace F by $10L$ and thus we will partition

$$16L+U$$

into three equal piles, starting with the $16L$.

We now carry down the six to get:

$$
\begin{array}{ccccc}
 & & 1 & & \\
3 &) & \overline{4} & \overline{6} & \overline{1} \\
 & & \underline{3} & \downarrow & \\
 & & 1 & 6 &
\end{array}
$$

Clearly we can 16 as

$$16 = 5 \cdot 3 + 1,$$

and thus each pile will have $5L's$ and there is one L remaining. Thus we obtain:

$$\begin{array}{cccc} & 1 & 5 & \\ 3\ ) & \overline{4} & \overline{6} & \overline{1} \\ & \underline{3} & & \\ & 1 & 6 & \\ & \underline{1} & \underline{5} & \\ & & 1 & \end{array}$$

Thus there is $F + 5L$ in each pile and there is $L + U$ remaining, still to be split up. We now write L as $10U$ and hence

$$L + U = 11U.$$

Symbolically, our long division thus far can be written as :

$$
\begin{array}{cccc}
 & 1 & 5 & \\
3 &)\overline{4} & \overline{6} & \overline{1} \\
 & \underline{3} & & \\
 & 1 & 6 & \\
 & \underline{1} & \underline{5} & \downarrow \\
 & & 1 & 1
\end{array}
$$

Using the Euclidean algorithm, we get

$$11 = 3 \cdot 3 + 2,$$

and thus:

$$
\begin{array}{rrrr}
 & 1 & 5 & 3 \\
3 &)\overline{4} & \overline{6} & \overline{1} \\
 & \underline{3} & & \\
 & 1 & 6 & \\
 & \underline{1} & \underline{5} & \\
 & & 1 & 1 \\
 & & & \underline{9} \\
 & & & 2
\end{array}
$$

Hence, we see that 461 can be spilt into three equal piles with $F + 5L + 3U$ blocks in each and $2U$ remaining. Thus we have

$$461 = 153 \cdot 3 + 2.$$

Let us also do a long division computation for

$$5739 \div 31.$$

Since the numbers are too big, we proceed without the visual aides, but we still justify all steps with the Euclidean algorithm.

Since 31 is a two digit number we will examine how many times 31 goes into 57, and this will tell how to split up 5700 into 31 equal pile. By the Euclidean algorithm, we see that

$$57 = 1 \cdot 31 + 26.$$

Symbolically, we represent this as:

$$\begin{array}{lccccc} & & & 1 & & \\ 31 &) & \overline{5} & \overline{7} & \overline{3} & \overline{9} \\ & & \underline{3} & \underline{1} & & \\ & & 2 & 6 & & \end{array}$$

We then carry down the 3 to get

$$\begin{array}{rcccc}
 & & 1 & & \\
31 &)\overline{5} & \overline{7} & \overline{3} & \overline{9} \\
 & \underline{3} & \underline{1} & & \\
 & 2 & 6 & 3 &
\end{array}$$

Now, applying the Euclidean Algorithm again, we get

$$263 = 8 \cdot 31 + 15$$

and thus

$$\begin{array}{cccccc} & & 1 & 8 & \\ 31 &)\overline{5} & \overline{7} & \overline{3} & \overline{9} \\ & \underline{3} & \underline{1} & & \\ & 2 & 6 & 3 & \\ & \underline{2} & \underline{4} & \underline{8} & \\ & & 1 & 5 & \end{array}$$

Carrying down the 9 we get:

$$
\begin{array}{rlllll}
 & & & 1 & 8 & \\
31 &) & \overline{5} & \overline{7} & \overline{3} & \overline{9} \\
 & & \underline{3} & \underline{1} & & \\
 & & 2 & 6 & 3 & \\
 & & \underline{2} & \underline{4} & \underline{8} & \\
 & & & 1 & 5 & 9
\end{array}
$$

Applying the Euclidean algorithm we get

$$159 = 5 \cdot 31 + 4$$

and thus

$$
\begin{array}{rlccc}
 & & 1 & 8 & 5 \\
31 &)\overline{5} & \overline{7} & \overline{3} & \overline{9} \\
 & \underline{3} & \underline{1} & & \\
 & 2 & 6 & 3 & \\
 & \underline{2} & \underline{4} & \underline{8} & \\
 & & 1 & 5 & 9 \\
 & & \underline{1} & \underline{5} & \underline{5} \\
 & & & & 4
\end{array}
$$

Hence, we see that

$$5739 = 185 \cdot 31 + 4.$$

Some Homework Exercises:

For the following, perform the long division algorithm to evaluate the expression, and express the larger number in terms of the smaller number using the Euclidean algorithm.

1. $732 \div 6$ **Answers:** 122; $732 = 122 \cdot 6$

2. $945 \div 9$ **Answers:** 105; $945 = 105 \cdot 9$

3. $947 \div 6$ **Answers:** 122 R 2; $945 = 105 \cdot 9 + 2$

4. $327 \div 5$ **Answers:** 65 R 2; $327 = 65 \cdot 5 + 2$

5. $327 \div 12$ **Answers:** 27 R 3; $327 = 27 \cdot 12 + 3$