Math 102: The Integers, Fractions, and the Rational Numbers

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The Set of Integers:

If we are willing to add elements to $\mathbb{W}$ to create the so-called **Integers**, we can use **subtraction** of whole numbers as a model. For instance, given any whole number b we can compute

$$a - b$$

as long as $a \geq b$, $a \in \mathbb{W}$. We may define ^{-}b to be the object that satisfies

$$a - b = a + (^{-}b)$$

for any whole number $a \geq b$. Thus, we see that ^{-}b is the unique object that satisfies

$$b + (^{-}b) = 0.$$

That is, given $b \in \mathbb{W}$ we would like to solve

$$b + x = 0$$

for some x. If we define x to be the solution to this equation, then we denote this x by ${}^{-}b$ or by $-b$ *.

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*For now we will use the notation ${}^{-}b$ instead of $-b$ to make it easier to distinguish between subtraction and negation.

Thus the set of **integers** is the set

$$\mathbb{Z} = \{\cdots {}^-4, {}^-3, {}^-2, {}^-1, 0, 1, 2, 34, \cdots\}$$

where here ${}^-a$ is the additive inverse of a, as described above. That is for a whole number a, ${}^-a$ is the unique integer that satisfies

$$a + ({}^-a) = 0.$$

Moreover, a is the additive inverse of ${}^-a$, since $a + ({}^-a) = 0$, that is

$${}^-({}^-a) = a.$$

Historically the need for negative numbers arose in trade and commerce. One may think of positive numbers as counting assets or surpluses and negative numbers as counting debts or losses. This model may be used to introduce addition of integers under the name of the **chip model** for addition, or the equivalent **charged particle** model for addition of integers.

Probably the first mathematicians to think of negative numbers in a somewhat modern sense – arising as solutions to algebra problems – were the Diophantus, a Greek mathematician, and then Brahmagupta, a Hindu mathematician. It wasn't until the 1500's or so that their ideas were rediscovered and somewhat accepted in Europe, some 1000 years later. For instance, the Italian mathematician Cardano was the first among European mathematicians to consider negative numbers as solutions to algebra problems, but saw these as "fictitious" or "absurd" numbers. Moreover, opposition to considering them as numbers existed in the mathematics community until at least the late 18^{th} century.

The Chip Model for Addition of Integers:

Recall the so-called comparison model for subtraction. This model compares two collections or sets with different numbers of elements.

For instance if $m, n \in \mathbb{W}$ with $n \geq m$ we may find a collection A with n elements in it and a collection B with m elements. If we were to find a one-to-one correspondence between B and some proper subset of A, how many things in A would not be paired with an element of B?

Of course the answer is $n - m$, or, if you wish, $n + (^{-}m)$.

We want to use this model to discuss subtraction for any two positive integers (natural numbers) a, b, or equivalently, the addition of any two integers which differ in sign. If we want to compute $a - b$, we find a collection A with a objects in it and another collection B with b objects in it. If $a \geq b$ we are done, since that was discussed above. Thus, we only need to consider the case where $a < b$.

Here we put the set A into one-to-one correspondence with a proper subset of B that has a elements in it. If we count how many objects that are left in B, we see that it is $b-a$. Since all the elements of A are paired with a of the elements of B, there are $b-a$ elements in B remaining that have not been paired with anything in A. We may think of the remaining $b-a$ elements as a deficit. Hence we can define

$$a-b =^{-} (b-a).$$

Thus we see that we can define

$$a-b = a + (^{-}b)$$

for any two whole numbers a, b.

Let $a, b \in \mathbb{W}$. If we take the equations

$$a + (^{-}a) = 0, \quad b + (^{-}b) = 0$$

and add both sides together, we see that

$$a + b + (^{-}a) + (^{-}b) = 0.$$

This gives us

$$^{-}(a + b) = (^{-}a) + (^{-}b).$$

Thus, in all cases we have defined what it means to add any two integers together, and thus as well, what it means two subtract any integer from any other integer.

The definition of addition of any two integers that is discussed above is essentially the idea behind the chip model of addition of integers. Here, positive integers may be thought of as a collection of black chips, and negative integers, as a collection of red chips. Adding integers corresponds with putting the two collections together, and removing pairs of red and black chips until only red or only black chips are remaining. How many red chips remaining or black chips remaining is the sum of the two integers.

The Number Line Model for Addition and Subtraction of Integers:

A visual model for this method will be discussed in lecture. Essentially it involves thinking of integers – or numbers in general – as positions along the number line, an infinitely long line with notches marked along it like a ruler.

Here we think of numbers as signed distances, which may be represented as arrows of various lengths, pointing in one of two directions. Here an arrow represents a number, and the head of an arrow is pointing to the left if the number is negative, and to the right if the number is positive. Addition of two numbers a and b corresponds with placing $b's$ arrow tail upon $a's$ arrow head. The resulting number $a + b$ has its arrow tail as $a's$ arrow tail and its arrow head as $b's$ arrow head (pointing in the appropriate direction).

Properties of Addition on the Set of Integers:

Let $a, b, c \in \mathbb{Z}$:

- **Closure property of addition:** $a + b$ is defined as an integer and is unique.
- **Commutativity of addition:**

$$a + b = b + a$$

- **Connection between addition and subtraction:**

$$a + ({}^{-}b) = a - b$$

- **Associative property of addition:**

$$a + (b + c) = (a + b) + c$$

This allows us to define summation for more that two addends, without ambiguity.

- **Identity property of addition:** There is a unique integer 0, the **additive identity**, which has the property
$$a + 0 = a = 0 + a$$
for all $a \in \mathbb{Z}$.

- **The distributive property of $-$:**
$$({}^{-}a) + ({}^{-}b) =^{-} (a + b)$$

- For any $a \in \mathbb{Z}$
$$a + ({}^{-}a) = ({}^{-}a) + a = 0$$
and thus
$${}^{-}({}^{-}a) = a.$$

The Absolute Value of an Integer:

The **absolute value** of an integer is defined to be the distance between the integer and 0 on the number line. Thus,

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ {}^{-}a & \text{if } a < 0 \end{cases}$$

For instance

$$|{}^{-}3| = 3, \quad |3| = 3$$

$$|0| = 0$$

$$|{}^{-}25| = 25, \quad |45| = 45$$

$$|3 - 2| = |2 - 3| = 1$$

Some Homework Problems:

1. Use both the chip model and number line model as visual aids in performing the following additions or subtractions:

   $$^{-}3+5, \quad 5-7, \quad ^{-}3-7, \quad 4+6$$

   Answer: $2;^{-}2;^{-}10;10$

2. Perform the following computations:

   $^{-}(^{-}2-6), \quad ^{-}|^{-}6|, \quad |5-6|, \quad 2-(5-7+6)$

   Answer: $8;^{-}6;1;^{-}2$

Multiplication of Integers:

We already know that the set of whole numbers is closed with respect to multiplication. We wish to extend the operation of multiplication from the set of whole numbers $\mathbb{W}$ to the set of integers $\mathbb{Z}$ in a way that the integers are closed under multiplication, and multiplication is associative. Moreover, we require that the multiplication $a \cdot b$ for whole numbers a, b agrees with our initial definition of multiplication of whole numbers

Given whole numbers a, b, we know what $a \cdot b$ is from our earlier discussions. The issue currently at hand is how should we define

$$({}^{-}a) \cdot b, \quad a \cdot ({}^{-}b), \quad and \quad ({}^{-}a) \cdot ({}^{-}b).$$

Once we define this, we shall know how to multiply any two integers.

To settle this, we will first show that

$$0 \cdot a = 0, \quad 0 \cdot (^-a) = 0, \quad (^-1) \cdot a =^- a$$

for any whole number a, and

$$(^-1) \cdot (^-1) = 1.$$

This will yield

$$(^-a) \cdot b = a \cdot (^-b) =^- (a \cdot b)$$

and

$$(^-a) \cdot (^-b) = a \cdot b$$

for any whole numbers a, b.

Proof of the above claims:

- $$0 \cdot a = 0 = 0 \cdot (^{-}a)$$

- $$(^{-}1) \cdot a =^{-} a$$

- $$(^{-}1) \cdot (^{-}1) = 1$$

Recall that for any integer

$$0 + c = c.$$

Thus, for any whole number a we have

$$0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a.$$

Subtracting $0 \cdot a$ from both sides yields

$$0 = 0 \cdot a.$$

Now,

$$a + ({}^{-}a) = a - a = 0$$

and thus

$$0 \cdot a + 0 \cdot ({}^{-}a) = 0 \cdot (a - a) = 0 \cdot 0 = 0.$$

However $0 \cdot a = 0$ and thus $0 \cdot ({}^{-}a)$ must also be 0.

By the above, we see

$$({}^{-}1) \cdot a + a$$

$$= ({}^{-}1) \cdot a + 1 \cdot a$$

$$= ({}^{-}1 + 1) \cdot a$$

$$= 0 \cdot a = 0.$$

Thus

$$(^{-}1) \cdot a + a = 0.$$

Adding ^{-}a to both sides yields

$$(^{-}1) \cdot a =^{-} a.$$

Likewise,

$$({}^{-}1) \cdot ({}^{-}1) + ({}^{-}1) = ({}^{-}1) \cdot ({}^{-}1 + 1) = 0.$$

Adding 1 to both sides yields

$$({}^{-}1) \cdot ({}^{-}1) = 1.$$

Hence, we see that

$$({}^{-}a) \cdot b = (({}^{-}1) \cdot a) \cdot b = ({}^{-}1) \cdot (a \cdot b) =^{-} (a \cdot b)$$

Likewise,

$$({}^{-}a) \cdot ({}^{-}b) = ({}^{-}1)^2 \cdot a \cdot b = a \cdot b.$$

Properties of Multiplication on the Set of Integers:

Let $a, b, c \in \mathbb{Z}$:

- **Commutativity:**

$$a \cdot b = b \cdot a$$

- **Associativity:**

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

- **Multiplication by 1:**

$$1 \cdot a = a = a \cdot 1$$

- **Multiplication by $^{-}1$:**

$$(^{-}1) \cdot a =^{-} a$$

- **Multiplication by 0:**

$$0 \cdot a = 0 = a \cdot 0$$

- **Distributive Property:**

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$a \cdot (b - c) = (a \cdot b) - (a \cdot b)$$

Some Homework Problems:

1. Perform the following computation, writing your answer as a single integer:

$$(^{-}2) \cdot 5$$

Answer: $^{-}10$

2. Perform the following computation, writing your answer as a single integer:

$$(^{-}2) \cdot (^{-}5)$$

Answer: 10

3. Perform the following computation, writing your answer as a single integer:

$$(^{-}2) \cdot 5 + (^{-}2) \cdot (^{-}3)$$

Answer: $^{-}4$

4. Perform the following computation, writing your answer as a single integer:

$$(^{-}2) \cdot 5 - (^{-}2) \cdot (^{-}3)$$

Answer: $^{-}16$

Ordering of the Integers:

We say that a is **less than** b, written as either

$$a < b \quad or \quad b > a,$$

if $b - a$ is a natural number.

We say that a is **less than or equal to** b, written as either

$$a \leq b \quad or \quad b \geq a,$$

if $b - a$ is a whole number.

Equivalently, if $a < b$, we may say b is **greater than** a, and if $a \leq b$, we may say b is **greater than or equal to** a.

Division of Integers: Fractions and the Rational Numbers

The idea of the existence of *numbers* between any two integers was developed by the Greek Pythagoreans in the 6^{th} century B.C.E. Before the Greeks, the Babylonians and Egyptians has used a few numbers we would call **fractions**, mainly those of the form $\frac{1}{n}$ with $n \in \mathbb{N}$, as well as a few other convenient ones such as $\frac{2}{3}$, etc.

The word **fraction** comes from the Latin *fractus*, which translates as broken. Hence we may initially consider fractions such as $\frac{2}{3}, \frac{1}{2}$, etc. as **broken numbers** or pieces of 1.

Given two natural numbers a, b with $b \neq 0$, the quantity

$$\frac{a}{b} = a \div b$$

may not be a natural number. Thus, given any two natural numbers, we still need to define what $\frac{a}{b}$ is in general.

We will start by defining

$$\frac{1}{b}$$

for $b \neq 0$ by using the **part-to-whole model**.

Once we have the fractions of the form

$$\frac{1}{b}$$

defined we may define

$$\frac{a}{b} = \underbrace{\frac{1}{b} + \frac{1}{b} + \cdots + \frac{1}{b}}_{a \quad times}$$

for any two whole numbers a, b with $b \neq 0$.

The fraction of the form $\frac{a}{b}$, with $a, b \in \mathbb{W}$, $b \neq 0$ is in general called a **Rational Number**. In the fraction $\frac{a}{b}$, the number a is called the **numerator** and the number b is called the **denominator**.

We define the number $\frac{1}{b}$, for $b \in \mathbb{N}$, by bisecting an object – the whole, which we may think of as the number 1 – into b equal pieces or parts. One of these parts we think of as $\frac{1}{b}$. In general, for $a \in \mathbb{N}$, we think of $\frac{a}{b}$ as a of these parts. At this point a visual model is very useful to illustrate the subdivision of the whole into equal parts.

We can define what it means to be a positive or negative rational number as well:

Given $a \in \mathbb{W}, b \in \mathbb{N}$ we define $^{-}(\frac{a}{b})$ to be the unique solution to the algebra problem

$$x + \frac{a}{b} = 0.$$

We then extend division onto the set of integers by

$$^{-}(\frac{a}{b}) = \frac{^{-}a}{b} = \frac{a}{^{-}b},$$

$$\frac{a}{b} = \frac{^{-}a}{^{-}b}.$$

Summarizing, this says:

$$\frac{positive}{positive} = positive,$$

$$\frac{positive}{negative} = negative,$$

$$\frac{negative}{positive} = negative,$$

$$\frac{negative}{negative} = positive.$$

Note that the positive rationals are the ones of the form $\frac{m}{n}$ with $m, n \in \mathbb{Z}$ where m, n have the same sign, and the negatives are the ones where m, n have different signs. 0 is neither positive nor negative.

Thus the set of rational numbers is the collection

$$\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$$

and this set is denoted by $\mathbb{Q}$.

We add, subtract, multiply and divide two rational numbers $\frac{m}{n}, \frac{a}{b}$ by

$$\frac{a}{b} \pm \frac{m}{n} = \frac{(a \cdot n) \pm (b \cdot m)}{b \cdot n},$$

$$\frac{a}{b} \cdot \frac{m}{n} = \frac{a \cdot m}{b \cdot n},$$

and

$$\frac{a}{b} \div \frac{m}{n} = \frac{a}{b} \cdot \frac{n}{m} = \frac{a \cdot n}{b \cdot m} \quad for \quad m \neq 0.$$

The Non-uniqueness of a Representation of a Rational Number:

Recall that a rational number is a number of the form

$$\frac{a}{b}, \quad a, b \in \mathbb{Z}, \quad b \neq 0.$$

By the way we defined a rational number, there is a non-uniqueness in the representation. That is, for any non-zero integer k, we have

$$\frac{a}{b} = \frac{k \cdot a}{k \cdot b}.$$

Thus

$$\frac{1}{2}, \quad \frac{-1}{-2}, \quad \frac{2}{4}, \quad \frac{-3}{-6}, \quad \frac{50}{100}, \quad \ldots$$

all represent the same rational number.

However, if the rational number is positive, we can always find a representation of a given rational number so that the greatest common divisor of the numerator and denominator is one. Such a fraction representation is said to be in **reduced form**. Thus, in the above example, $\frac{1}{2}$ is in reduced form.

Some Examples:

$$\frac{3}{5} + \frac{2}{7} = \frac{3 \cdot 7 + 2 \cdot 5}{5 \cdot 7} = \frac{31}{35}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{1 \cdot 3 - 1 \cdot 2}{2 \cdot 3} = \frac{1}{6}$$

$$\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$$

$$\frac{2}{3} \div \frac{7}{4} = \frac{2}{3} \cdot \frac{4}{7} = \frac{2 \cdot 4}{3 \cdot 7} = \frac{8}{21}.$$

Mixed Numbers:

A rational number $\frac{a}{b}$ is called a **proper fraction** if $|a| \leq |b|$, and otherwise it is called an **improper fraction.**

Often one wishes two express an improper fraction

$$\frac{a}{b}$$

as a **mixed number**, which is an integer plus a proper fraction. To do this, we use the Euclidean Algorithm (in the case where $0 < b < a$) to write

$$a = q \cdot b + r$$

and thus

$$\frac{a}{b} = q + \frac{r}{b}$$

which is often written

$$q\frac{r}{b}.$$

Example:

Consider $\frac{121}{13}$. By the Euclidean Algorithm, we write

$$121 = 9 \cdot 13 + 4$$

and thus

$$\frac{121}{13} = 9 + \frac{4}{13} = 9\frac{4}{13}.$$

Decimal Representation of Non-negative Proper Fractions:

Let d_i represent a digit, that is a number in the collection

$$\{0,1,2,3,4,5,6,7,8,9\}.$$

Then we use the notation

$$0.d_1d_2d_3\cdots d_n$$

to represent the number

$$\frac{d_1d_2d_3\cdots d_n}{10^n},$$

where

$$10^n = \underbrace{100\cdots0}_{n\quad 0's}.$$

This number also equals

$$\frac{d_1}{10}+\frac{d_2}{100}+\frac{d_3}{1000}+\cdots+\frac{d_n}{10^n}.$$

For instance

$$0.123 = \frac{123}{1000} = \frac{1}{10} + \frac{2}{100} + \frac{3}{1000}.$$

Also,

$$\frac{1}{2} = \frac{5}{10} = 0.5.$$

Order and the Rationals:

A rational number $\frac{a}{b}$, with $b > 0$ is greater than (or equal to) another rational number $\frac{n}{m}$, with $m > 0$, if

$$a \cdot m > b \cdot n \quad (a \cdot m \geq b \cdot n).$$

Thus,

$$\frac{3}{4} > \frac{2}{5}$$

since

$$15 = 3 \cdot 5 > 2 \cdot 4 = 8.$$

The Density of the Rational Numbers:

If you were to make a notch along the number line for each rational number, you would find out that the entire number line looks as if it has notches everywhere. This property of the rational numbers is a so-called density property.

Essentially you can see why this is true by observing that given two rational numbers, there exists another rational number in between the two given numbers – one such number is their average. That is

$$\frac{a}{b} < \frac{n}{m}$$

implies

$$\frac{a}{b} < \frac{a \cdot m + b \cdot n}{2 \cdot m \cdot b} < \frac{n}{m}.$$

For example, between $\frac{2}{3}$ and $\frac{3}{4}$ on the number line is the number $\frac{17}{24}$.

Some Facts about Rational Numbers and their Decimal Representations:

Every rational number when expressed as a decimal number will be either a terminating or repeating decimal number.

Moreover, a decimal number that either terminates or has a repeating pattern must be a *rational number*.

The Number Line and the Set of Real Numbers:

If one thinks of the number line as a collection of all possible positions along an infinitely long ruler, with 0 at some specific point, and the positive numbers to the right of 0, and the negatives to the left, one sees that the rational numbers are dense. That is, the rational numbers are everywhere along this number line. No matter how small of a piece of the number line one chooses, he/she can find a rational number in this piece.

However, by the above characterization of the rationals by there decimal representations, and that the collection of all decimal representations may be thought of as the set of real numbers as well, one sees that there are numbers missing, namely the so-called **irrational numbers**.

For example, the number

$$0.1010010001000010\cdots$$

must be the decimal expansion of an irrational number. Even though there is a pattern here, it is not a repeating patter. Moreover, the expansion does not terminate, since there are infinitely many ones in this decimal expansion.

Some Examples:

$$0.33333\cdots = \frac{1}{3}.$$

Let us see why. Let $x = 0.333\cdots$, then $10x = 3.333\cdots$. Thus $9x = 10x - x = 3$, which yields

$$x = \frac{3}{9} = \frac{1}{3}.$$

$$0.3333\cdots$$

is often written as

$$0.\overline{3}.$$

$$0.\overline{123}$$

means

$$0.123123123123123\cdots$$

To write this as a rational number we let $x = 0.\overline{123}$ and observe that

$$1000x = 123.\overline{123}$$

and thus

$$999x = 123.$$

This gives us

$$x = \frac{123}{999}.$$

Writing Rational Numbers as Decimals:

Here we shall proceed by example, simply noting that

$$\frac{a}{b}$$

must either terminate or repeat. If it repeats, it must have a repeating pattern no longer than $b - 1$ digits long.

To find a decimal representation for $\frac{a}{b}$, we perform a long division

$$\text{b} \quad)\overline{a.000000}$$

This we shall carry out in lecture.

Some Homework Exercises:

1. Put the following rational numbers in order, from smallest to largest
$$-\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{-2}, \frac{15}{4}$$
Answer: $\frac{3}{-2}, -\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{15}{4}$

2. Perform the following computations, writing your answer as a single fraction in reduced form, and if necessary as a mixed number:
$$\frac{1}{2} - \frac{3}{7}, \quad \frac{2}{3} + \frac{5}{7}, \quad \frac{3}{4} \cdot \frac{5}{7}, \quad \frac{3}{4} \div \frac{5}{2}$$
Answers: $\frac{1}{14}$; $1\frac{8}{21}$; $\frac{15}{28}$; $\frac{3}{10}$.

3. Find a decimal representation for the fraction $\frac{1}{7}$. **Answer** $0.\overline{142857}$

4. Find a fraction (in reduced form) that equals

$$0.125$$

Answer: $\frac{1}{8}$.

5. Find a fraction (in reduced form) that equals

$$0.\overline{125}$$

Answer: $\frac{125}{999}$.

6. Given an example of an irrational number other than

$$0.10100100010\cdots$$

using a decimal expansion.