Math 102: The Set of Real Numbers and Decimal Representations

Dr. Richard Mikula

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Arithmetic with Decimals:

We may perform arithmetic with decimals which have terminating decimal expansions by seeing them as fractions or rational numbers. For example:

$$0.45 + 0.361 = \frac{45}{100} + \frac{361}{1000}$$

$$= \frac{450}{1000} + \frac{361}{1000} = \frac{450 + 361}{1000}$$

$$= \frac{811}{1000} = 0.811.$$

This may be written symbolically as

$$\begin{array}{cccccc} & 0 & . & 4 & 5 & \\ + & 0 & . & 3 & 6 & 1 \\ \hline \end{array}$$

We then add extra zeros if necessary so that each decimal has the same number of terms to the right of the decimal point:

$$\begin{array}{cccccc} & 0 & . & 4 & 5 & 0 \\ + & 0 & . & 3 & 6 & 1 \\ \hline \end{array}$$

Then we add as if we are adding whole numbers:

$$\begin{array}{cccccc} & & & 1 & & \\ & 0 & . & 4 & 5 & 0 \\ + & 0 & . & 3 & 6 & 1 \\ \hline & 0 & . & 8 & 1 & 1 \end{array}$$

Subtraction in done in a similar manner:

$$0.5 - 0.23 = \frac{5}{10} - \frac{23}{100}$$

$$= \frac{50}{100} - \frac{23}{100} = \frac{50 - 23}{100}$$

$$= \frac{27}{100} = 0.27.$$

Symbolically, this may be done by:

$$\begin{array}{cccccc} & 0 & . & 5 & \\ - & 0 & . & 2 & 3 \\ \hline \end{array}$$

Adding zeros if necessary, so that each decimal has the same number of terms to the right of the decimal point, we get:

$$\begin{array}{ccccc} & 0 & . & 5 & 0 \\ - & 0 & . & 2 & 3 \\ \hline \end{array}$$

Then, the arithmetic is done as if we were doing it with whole numbers:

$$\begin{array}{ccccc} & & & \searrow 4 & \searrow 10 \\ & 0 & . & 5 & 0 \\ - & 0 & . & 2 & 3 \\ \hline & 0 & . & 2 & 7 \end{array}$$

Multiplication may be done in a similar manner:

$$2.31 \times 0.456 = \frac{231}{100} \cdot \frac{456}{1000}$$

$$= \frac{231 \cdot 456}{100 \cdot 1000}$$

$$= \frac{105,336}{100,000}$$

$$= 1.05336$$

Symbolically, this may be done by:

$$\begin{array}{cccccc} & 2 & . & 3 & 1 & \\ \times & 0 & . & 4 & 5 & 6 \\ \hline \end{array}$$

Here it is not necessary that the decimal places are lined up, we simply proceed as it were multiplication of whole numbers:

$$\begin{array}{cccccc} & & 2 & . & 3 & 1 \\ \times & 0 & . & 4 & 5 & 6 \\ \hline \end{array}$$

We shall do this in steps, noting that the decimal places are not relevant until the final step.

$$\begin{array}{cccccc} & & {}^{1} & & & \\ & & 2 & . & 3 & 1 \\ \times & 0 & . & 4 & 5 & 6 \\ \hline & & 1 & 3 & 8 & 6 \end{array}$$

		1			
		↘1			
		2	.	3	1
×	0	.	4	5	6
		1	3	8	6
	1	1	5	5	

		1			
		↘1			
		↘1			
		2	.	3	1
×	0	.	4	5	6
		1	3	8	6
	1	1	5	5	
	9	2	4		

$$
\begin{array}{cccccc}
 & & 1 & & & \\
 & & \searrow 1 & & & \\
 & & \searrow 1 & & & \\
 & & 2 & . & 3 & 1 \\
\times & 0 & . & 4 & 5 & 6 \\
\hline
 & & {\scriptstyle 1} & {\scriptstyle 1} & & \\
 & & 1 & 3 & 8 & 6 \\
 & 1 & 1 & 5 & 5 & \\
+ & 9 & 2 & 4 & & \\
\hline
1 & 0 & 5 & 3 & 3 & 6
\end{array}
$$

At this point we count the number of decimal places to the right of the decimal point that appear in total in our multiplication problem

$$3.31 \times 0.456,$$

in this case 5. Then we take our number

$$105336.$$

and move the decimal point we placed after it 5 places to the left, thus resulting in

$$1.05336,$$

which is the desired product.

Lastly, we shall discuss division of numbers which are decimals. We shall proceed by examining an example:

$$12.456 \div 1.79$$

which would be computed symbolically by

$$1.79 \quad)\overline{12.456}$$

or by

$$\frac{12.456}{1.79}.$$

However, we may multiply any fraction by any non-zero number over itself and the value remains unchanged. It we multiply the denominator by 100, we get a whole number. Thus we shall multiply the fraction by 100 over itself.

Thus, we get

$$\frac{12.456}{1.79} \cdot \frac{100}{100} = \frac{1245.6}{179}$$

which is the same division problem as

$$179 \enspace)\overline{1245.6} \enspace .$$

We then proceed as if this was any other long division, just moving the decimal place up first to get:

$$179 \enspace)\overset{.}{\overline{1245.6}} \enspace .$$

First, we look at 179 and 1245, and ask ourselves what is the largest multiple of 179 that is less than or equal to 1279? In this case, it is 6. That is

$$1245 = 6 \cdot 179 + 171.$$

Thus, we get:

$$\begin{array}{rr} & 6. \\ 179 &)\overline{1245.6} \\ & \underline{-1074.}\downarrow \\ & 171.6 \end{array}.$$

We then apply the Euclidean algorithm to 179 and 1716, getting

$$1716 = 9 \cdot 179 + 105 = 1611 + 105,$$

and thus

$$\begin{array}{rr} & 6.9 \\ 179 &)\overline{1245.60} \\ & \underline{-1074.0}\downarrow \\ & 1716\downarrow \\ & \underline{-1611}\downarrow \\ & 105.0 \end{array} .$$

We then apply the Euclidean algorithm to 179 and 1050, getting

$$1050 = 5 \cdot 179 + 155 = 895 + 155,$$

and thus

$$\begin{array}{rl} & 6.95 \\ 179 & \overline{)1245.600} \\ & \underline{-1074.0} \downarrow \\ & 1716 \downarrow \\ & \underline{-1611} \downarrow \\ & 1050 \downarrow \\ & \underline{-895} \downarrow \\ & 155.0 \end{array} .$$

We continue in this manner until the Euclidean algorith returns a remainder of 0, until we see a repeat in the remainder, or until we have obtained the desired quotient to enough decimal places. It turns out that

$$12.456 \div 1.78 = 6.95865921787709497 2067 \cdots$$

We know that this quotient is indeed a rational number, but it could have a repeating pattern that contains up to 178 terms, since we were applying the Euclidean algorithm with 179 as the smaller number each time.

Some Homework Exercises:

Perform the following calculations:

1.
$$0.456 + 1.67$$
Answer: 2.126

2.
$$1.689 - 0.89$$
Answer: 0.799

3.
$$0.456 \times 1.69$$
Answer: 0.77064

4.

$$2.3 \div 1.61$$

Answer: $1.\overline{428571}$

Percents:

The word **percent** comes from the Latin *per centum*, which means, per hundred. Thus x percent, often written $x\%$ is really $\frac{x}{100}$. Thus 5 percent of 42 translates as

$$\frac{5}{100} \times 42 = 0.05 \cdot 42 = 2.1.$$

Likewise, if you wanted to know what percent 4 is of 15 you would solve the problem

$$\frac{x}{100} \cdot 15 = 4$$

for x. The solution being

$$\frac{x}{100} = \frac{4}{15} = 0.2\overline{6}$$

or

$$x = 26.\overline{6}\%.$$

Some Homework Problems:

1. 12 is what percent of 75? **Answer:** 16

2. Find 52% of 87. **Answer:** 45.24.

Some Facts about Rational Numbers and Their Decimal Representations:

Every rational number when expressed as a decimal number will be either a terminating or repeating decimal number.

Moreover, a decimal number that either terminates or has a repeating pattern must be a *rational number*.

The Number Line and the Set of Real Numbers:

If one thinks of the number line as a collection of all possible positions along an infinitely long ruler, with 0 at some specific point, and the positive numbers to the right of 0, and the negatives to the left, one sees that the rational numbers are dense. That is, the rational numbers are everywhere along this number line. No matter how small of a piece of the number line one chooses, he/she can find a rational number in this piece.

However, by the above characterization of the rationals by there decimal representations, and that the collection of all decimal representations may be thought of as the set of **real numbers** $\mathbb{R}$ as well, one sees that there are numbers missing, namely the so-called **irrational numbers**.

For example, the number

$$0.1010010001000010\cdots$$

must be the decimal expansion of an irrational number. Even though there is a pattern here, it is not a repeating patter. Moreover, the expansion does not terminate, since there are infinitely many ones in this decimal expansion.

The Irrational Numbers:

It was the ancient Greek mathematician Pythagoras (585-500 B.C.E.) who observed that every real number (i.e. position on the number line) cannot possibly be rational. Let us see why. To do this we must look at a right triangle, with sides each of length 1, and hypotenuse of length h. (Draw a picture).

The Pythagorean theorem tells us that a right triangle with sides of lengths a, b and hypotenuse of length c must satisfy

$$a^2 + b^2 = c^2.$$

Thus, we see that h satisfies

$$1^2 + 1^2 = h^2$$

or simply

$$h^2 = 2.$$

The mysterious h is then denoted by $h = \sqrt{2}$, since $h^2 = 2$.

We know that this number exists in the sense of a position on the number line since we can draw a line segment of this length h (see the right triangle again!).

Let us suppose that $h = \sqrt{2}$ was a rational number. Then we know that there are integers m, n so that

$$\sqrt{2} = \frac{m}{n}.$$

Moreover, we may suppose that the GCD of m, n is 1. Note that this is guaranteed by the fundamental theorem of arithmetic.

Squaring both sides of $\sqrt{2} = \frac{m}{n}$ yields

$$2 = \frac{m^2}{n^2}$$

and thus

$$2n^2 = m^2$$

which tells us that $2|m^2$. Now the fundamental theorem of arithmetic guarantees that 2 then must also divide m and hence $m = 2k$ for some integer k. Substituting this in for m yields

$$2n^2 = (2k)^2$$

and thus

$$n^2 = 2k^2.$$

By similar reasoning, $2|n$. This is our contradiction! (Why?)

Hence, $\sqrt{2}$ cannot be rational.

The Decimal Expansions of the Irrational Numbers:

From our observation that the rational numbers must have terminating or repeating decimal expansions, and all numbers that have terminating or repeating decimal expansions must be rational, we easily see the following:

An irrational number is a number that has neither a terminating nor a repeating decimal expansion.

Some other famous irrational numbers:

$$e, \pi$$

Here π is the ratio of a circumference of a circle to its diameter. That is

$$\pi = \frac{circumerence}{diameter}.$$

The decimal expansion of π is

$$\pi = 3.14159265358979323846 \cdots$$

e is the famous irrational number that comes up in so-called exponential growth and decay problems. If we define $n!$ by

$$n! = n \cdot (n-1) \cdot 3 \cdot 2 \cdot 1$$

for any whole number n, we see that $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, etc. Then e is given by

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{31} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{!}{7!} + \cdots$$

Moreover, e has the decimal expansion

$$e = 2.718281828459045235\cdots$$