Math 107 Sample Exam Number 2, November 2009

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1 Empirical Probability Questions

The nature of probability is summarized by the **law of large numbers.** This states that probability statements apply in practice to a large number of trials of an experiment, not on a single trial. It is the relative frequency over the long run that is accurately predictable, not individual events or precise totals.

1. An apartment complex has performed a survey on number of residents. The table below lists the data. Suppose that a random apartment is selected. Fill in the table to answer the following questions:

   (a) What is the probability of the apartment having at least 4 people living in it?

   (b) What is the probability that the apartment has at most 2 people living in it?

   (c) What is the probability that the apartment has at least 2 people living in it?

Number of People	Frequency	empirical probability
1	8	
2	14	
3	7	
4	12	
5	3	
6	1	

First we fill out the table, getting:

Number of People	Frequency	empirical probability
1	8	$\frac{8}{45} \approx 0.178$
2	14	$\frac{14}{45} \approx 0.311$
3	7	$\frac{7}{45} \approx 0.156$
4	12	$\frac{12}{45} \approx 0.267$
5	3	$\frac{3}{45} \approx 0.067$
6	1	$\frac{1}{45} \approx 0.022$

The answer to (a) is: $0.267 + 0.067 + 0.022$

The answer to (b) is: $0.178 + 0.311$

The answer to (c) is: $1 - 0.178$ or simply $0.311 + 0.156 + 0.267 + 0.067 + 0.022$

2. Incoming students to a university take a mathematics exam, consisting of 75 multiple choice exercises. Suppose the following table contains the distribution of scores on the exam. Please fill in the table, and answer the following questions:

Score	Number of students	empirical probability
0-15	2	
16-25	0	
26-35	8	
36-45	36	
46-55	110	
56-65	78	
66-75	66	

A random incoming student is selected.

(a) Find the probability that the student has scored between 56-75.

(b) Find the probability that the student has scored lower than 56.

(c) Find the probability that the student has scored higher than 25.

We fill in the table to get

Score	Number of students	empirical probability
0-15	2	$\frac{2}{300} \approx 0.007$
16-25	0	0
26-35	8	$\frac{8}{300} \approx 0.027$
36-45	36	$\frac{36}{300} = 0.12$
46-55	110	$\frac{110}{300} \approx 0.367$
56-65	78	$\frac{78}{300} = 0.26$
66-75	66	$\frac{66}{300} = 0.22$

The answer to (a) is: $0.26 + 0.22$.

The answer to (b) is: $0.007 + 0.027 + 0.12 + 0.367$.

The answer to (c) is: $0.027 + 0.12 + 0.367 + 0.26 + 0.22$.

2 Other Questions on Probability

1. For the sample space

$$S = \{a, b, c, d\},$$

do the following tables define a probability distribution? Please explain.

(a)

outcome	probability
a	0
b	0
c	1
d	0

(b)

outcome	probability
a	-0.001
b	0
c	0.001
d	1

(c)

outcome	probability
a	0.26
b	0.20
c	0.23
d	0.29

(d)

outcome	probability
a	0.3
b	0.3
c	0.2
d	0.25

(e)

outcome	probability
a	0.25
b	0.2
c	0.15
d	0.4

Answers: Yes, No, No, No, Yes.

2. Suppose that for an experiment with sample space S let A, B be 2 events. Answer the following questions:

   If $P(A) = 0.49, P(A \cap B) = 0.27$ and $P(A \cup B) = 0.89$ Find $P(B)$, $P(B^c)$, $P((A \cap B)^c)$ and $P(A^c \cup B)$.

   Answers (Draw a Venn Diagram to see this!):

   $$P(B) = 0.67, \quad P(B^c) = 0.33, \quad P((A \cap B)^c) = 1 - 0.27 = 0.73, \quad P(A^c \cup B) = 0.78.$$

3. Let an experiment consist of rolling a "fair" 20 sided die whose sides are numbered 1 through 20 and rolling a "fair" 12 sided die whose faces are labeled $-2, -4, \cdots, -24$.

   (a) Find the probability of getting an even number on the first die and a multiple of 6 on the second die.

   (b) Find the probability that the sum of their faces is 4.

   (c) Find the probability that the sum of their faces is positive.

   For this problem the sample space has $20 \cdot 12 = 240$ elements in it; each outcome is equally likely.

   The answer to (a) is: $\frac{40}{240}$. To see this note that there are 10 even numbers on the first die and 4 multiples of 6 on the second die.

   The answer to (b) is: $\frac{8}{240}$. to see this note that the ordered pairs that sum to 4 are: (6,-2), (8,-4), (10,-6), (12, -8), (14, -10), (16, -12), (18, -14), (20, -16).

   The answer to (c) is: $\frac{90}{240}$. This one is harder!

   If the second die is a -2, the first can be any number from 3 to 20.
   If the second die is a -4, the first can be any number from 5 to 20.
   If the second die is a -6, the first can be any number from 7 to 20.
   If the second die is a -8, the first can be any number from 9 to 20.
   If the second die is a -10, the first can be any number from 11 to 20.
   If the second die is a -12, the first can be any number from 13 to 20.
   If the second die is a -14, the first can be any number from 15 to 20.
   If the second die is a -16, the first can be any number from 17 to 20.
   If the second die is a -18, the first can be any number from 19 to 20.
   Counting the number of possible outcomes gives 90.

4. Let an experiment consist of rolling a "fair" 8 sided die whose sides are numbered 1 through 8 and rolling a "fair" 4 sided die whose faces are labeled $-2, -4, -6, -8$.

(a) Find the probability of getting an even number on the first die and a multiple of 6 on the second die.

(b) Find the probability that the sum of their faces is 4.

(c) Find the probability that the sum of their faces is positive.

For this problem the sample space has $8 \cdot 4 = 32$ elements in it; each outcome is equally likely.

These you can list easily:

$$\begin{gathered}(1,-2),(1,-4),(1,-6),(1,-8)\\(2,-2),(2,-4),(2,-6),(2,-8)\\(3,-2),(3,-4),(3,-6),(3,-8)\\(4,-2),(4,-4),(4,-6),(4,-8)\\(5,-2),(5,-4),(5,-6),(5,-8)\\(6,-2),(6,-4),(6,-6),(6,-8)\\(7,-2),(7,-4),(7,-6),(7,-8)\\(8,-2),(8,-4),(8,-6),(8,-8)\end{gathered}$$

For (a): note that the only outcomes in this event are $(2,-6),(4,-6),(6,-6),(8,-6)$. Thus the prob is $\frac{4}{32}$.

For (b): note that only (6,-2) and (8,-4) sum to 4. thus the probability is $\frac{2}{32}$.

For (c): the prob is $\frac{12}{32}$.

3 Conditional Probability Exercises

(a) Suppose A, B are two events and they have probabilities

$$P(A) = 0.5, \quad P(B) = 0.4$$

and

$$P(A \cap B) = 0.2$$

are A, B independent? What about if $P(A \cap B) = 0.21$?

In the first case yes, since $P(A) \cdot P(B) = 0.2 = P(A \cap B)$. In the second case, no.

(b) Given that two events A, B are independent, and have probabilities

$$P(A) = 0.65, \quad P(B) = 0.45$$

find

$$P(A \cap B), \quad and \quad P(A \cup B).$$

Because A, B are independent, we get $P(A \cap B) = 0.65 \cdot 0.45$ Thus

$$P(A \cup B) = 0.65 + 0.45 - 0.65 \cdot 0.45.$$

(c) Suppose $P(A) = 0.5$, $P(B) = 0.45$ and $P(A \cap B) = 0.35$, find $P(A|B)$, $P(B|A)$, $P(A \cup B)$. Are A and B independent?

No, the events are not independent. To see this notice

$$0.5 \cdot 0.45 = P(A) \cdot P(B) \neq P(A \cap B) = 0.35.$$

$$P(A \cup B) = 0.5 + 0.45 - 0.35.$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.35}{0.5} = 0.7.$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.35}{0.45}.$$

(d) In a math course suppose that 30 percent of the students failed the first exam, 15 percent failed the second exam and 8 percent failed both exams. Are the events failing the first exam and failing the second exam independent? Also find the probability of failing the second exam, given that the student failed the first.

Answer: $0.3 \cdot 0.15 \neq 0.08$, so they are dependent. $\frac{0.08}{0.3} \approx 0.26$.

4 Random Variables

1. An insurance company sells a 100,000 dollar policy it promises to pay the policy holder in the event they are disabled and can no longer work. The probability that a policy holder will be paid for loss of job due to medical reasons is 1 in 500. Should the insurance company expect to earn a profit if it sells the policies for 250 dollars each?

   Answer: Let X be the random variable that measures how much the insurance company brings in on a policy. Consider the table:

x	$f(x)$	$x \cdot f(x)$
-99750	0.002	-199.5
250	0.998	249.5

Summing the right column, we get

$$E(X) = 50.$$

Thus the insurance company will bring in, on average, 50 dollars per policy.

2. Let X be a random variable for an experiment. Suppose that X has the following probability density function:

x	$f(x)$
-1	0.9
0	0.05
10	0.04
100	0.01

Find $E(X)$ and $sd(X)$.

Answers:

x	$f(x)$	$x \cdot f(x)$	$x - E(X)$	$(x - E(X))^2$	$(x - E(X))^2 \cdot f(x)$
-1	0.9	-0.9	-1.5	2.25	2.025
0	0.05	0	-0.5	0.25	0.0125
10	0.04	0.4	9.5	90.25	3.61
100	0.01	1	99.5	9900.25	99.0025
SUM		$E(X) = 0.5$			104.65

Thus $E(X) = 0.5$ and $sd(X) = \sqrt{104.65} \approx 10.2298583$.

3. Suppose the random variable X defined in the previous problem represents the winnings or losses upon playing a game of chance. Is this game fair? If you were asked if you wanted to play it, would you choose to do so often? Also, Find $P(win \quad money)$, $P(loose \quad money)$

Answer: No, since $E(X) \neq 0$. Since $E(X) > 0$ you should play it as much as possible. $P(win \quad money) = 0.05$ and $P(loose \quad money) = 0.9$.

4. Suppose 25 percent of the items produced by a factory are defective. Suppose that 6 are chosen at random. Let X be the number of defective items in the collection of 6 that were chosen. Find:

   (a) $P(X = 2)$

   (b) $P(X < 2)$

(c) $P(X > 2)$

(d) $P(X \geq 2)$

(e) $E(X)$

(f) $sd(X)$

Answers: 0.2966309, 0.5339355, 0.1694336, 0.4660644, 1.5, 1.0606066.

5. If for the above factory, 100 items are chosen. Again letting X be the number of defective items, use Chebyshev's inequality to find a lower bound for

$$P(10 < X < 40).$$

Answer: Here $E(X) = 25, sd(X) = \sqrt{18.75} \approx 4.33$. Thus

$$P(10 < X < 40) \geq 1 - \frac{1}{(\frac{15}{4.33})^2} = 1 - \frac{1}{12}$$