Math 107: An Introduction to Probability

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The Nature of Probability

First, we must define some needed terminology in order to discuss chance and probability in a mathematically meaningful way:

An **experiment** is a controlled operation that yields a set of results.

The possible results of an experiment are called its **outcomes.**

An **event** is a subset of the set of outcomes of an experiment. *Note that the set of outcomes is sometimes called the* **Sample Space** *of an experiment. The set of outcomes of an experiment is our universal set.*

Probability is classified as either **empirical** (experimental) or **theoretical** (mathematical).

Empirical probability is the relative frequency of occurrence of an event, and is determined by actual observations of an experiment.

Theoretical probability is determined through a study of the possible outcomes that can occur for the given experiment.

We will indicate the probability of an event E by

$$P(E),$$

which is read "Probability of E."

The probability of an event is always a number between 0 and 1. It is 0 if it never occurs, and 1 if it definitely occurs.

Example: Suppose that a coin is flipped 25 times. Let H represent the outcome of a head, and T the outcome of a tail. Suppose the following were the outcomes of the 25 flips:

$$H, H, T, T, T, T, T, T, H, H, H, H,$$

$$H, H, T, T, H, H, H, T, H, T, T, H, T$$

The empirical probability of a head is the number of heads that occurred over the number of observations, that is

$$P(H) = \frac{13}{25}$$

and the empirical probability of a tail is the number of tails that occurred oner the number of outcomes, that is

$$P(T) = \frac{12}{25}.$$

Example: Suppose that two dice are rolled, and the outcomes are recorded:

$$(1,6),(1,2),(5,4),(6,5),(1,3)$$

$$(4,5),(4,3),(5,3),(2,1),(3,5)$$

$$(4,4),(2,5),(1,3),(6,3),(3,3)$$

We recorded the sum of the two faces:

$$7,3,9,11,4,9,7,$$

$$8,3,8,8,7,4,9,6$$

What is the empirical probability of 6, 7, and 8?

$$P(6)=\frac{1}{15},P(7)=\frac{3}{15},P(8)=\frac{3}{15}.$$

Exercise: Find the empirical probability of a 3, 5, and 9.

The following table represents the outcomes of tossing a "fair coin." this is a coin which has no bias. Thus, you would assign a theoretical probability to the outcome of H to be $\frac{1}{2}$.

No. Tosses	Exp. H's	Obs. H's	Rel. freq.
10	5	4	$\frac{4}{10}$
100	50	45	$\frac{45}{100}$
1,000	500	546	$\frac{546}{1000}$
10,000	5,000	4,852	$\frac{4852}{10000}$
100,000	50,000	49,770	$\frac{49770}{100000}$

Observe that in the previous table, the relative frequency is the empirical probability. Notice that it is close to $\frac{1}{2} = 0.5$.

The nature of probability is summarized by the **law of large numbers.** This states that probability statements apply in practice to a large number of trials, not to a single trial. It is the relative frequency over the long run that is accurately predictable, not individual events or precise totals.

Problem: The following table represents the number of active-duty military personnel by rank in the four major branches of the military as of Feb. 2005.

	Officers	Enlisted
Army	80,580	409,410
Navy	53,428	310,167
Air Force	73,331	159,164
Marines	18,893	287,328

Homework:

An active-duty person is selected at random. Determine the probability that the individual is

1. not an officer

2. in the army

3. in the army, but not an officer

4. not an officer, given that he/she is in the army.

Answers: 1. $\frac{1{,}166{,}069}{1{,}392{,}301}$, 2. $\frac{489{,}990}{1{,}392{,}301}$, 3. $\frac{409{,}410}{1{,}392{,}301}$, 4. $\frac{409{,}410}{489{,}990}$

The following table gives the number of robbery offenses for various robberies in 2003:

Type of robbery	freq. (in 1000's)	rel. freq.
Street or highway	131	
Commercial	61	
Gas Station	10	
Convenience store	26	
Residence	41	
Bank	7	

Homework:

1. Find the empirical probabilities, assuming that any robbery in 2003 is represented in one of those categories.
   Answer: 0.4746, 0.2210, 0.0362, 0.0942, 0.1486, 0.0254

2. Assuming a person was robbed in 2003, what is the probability that this occurred in either a residence or on the street or highway?
   Answer: 0.6232

3. Assuming a person was robbed in 2003, what is the probability that this didn't occur at a convenience store ?
   Answer: 0.9058

Theoretical Probability:

If each outcome of an experiment has the same chance of occurring as any other outcome, they are said to be **equally likely outcomes.**

If an experiment has equally likely outcomes, the probability of an event E is given by

$$P(E) = \frac{number \quad of \quad outcomes \quad in \quad E}{total \quad number \quad of \quad outcomes \quad in \quad experiment}.$$

Moreover, in any experiment, the sum of all the probabilities of all the outcomes must add up to 1.

Examples: Suppose you roll a die, find the probability of rolling a 5. Find the probability of rolling at least a 5.

Solution: All outcomes are equally likely (if it is a fair die), Thus

$$P(5) = \frac{1}{6},$$

$$P(at \quad least \quad 5) = P(5) + P(6) = \frac{1}{6} + \frac{1}{6}.$$

Example: Suppose you roll two "fair" dice. You record the sum of the faces. Find the probability that you have **a:** 7, **b:** 5, **c:** at least a 10.

The possible outcomes for rolling two fair dice we can think of as ordered pairs (x, y). There are 36 possible outcomes, which are:

$$(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)$$

$$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$$

$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$$

$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$$

$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)$$

$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)$$

The outcomes which add up to 7 are

$$(1,6),(6,1),(2,5),(5,2),(3,4),(4,3),$$

each of which are equally likely. Thus the probability of rolling a total of 7 is $\frac{6}{36}$.

The outcomes which add up to totals of at least 10, must have totals of $10, 11$ or 12. The outcomes for 10 are

$$(4,6),(6,4),(5,5),$$

the outcomes for 11 are

$$(5,6),(6,5)$$

and the outcome for 12 is

$$(6,6).$$

Thus, the probability of rolling a total of at least 10 is

$$\frac{6}{36}.$$

Some general facts to keep in mind:

- The probability of an event that cannot occur is 0.

- The probability of an event that must occur is 1.

- Probabilities of events are always numbers between 0 and 1.

- The sum of the probabilities of all the outcomes in the Sample Space of an experiment must add up to 1.

- If E is an event ($E \subseteq S$), the complementary event E^c is the event that consists of all other outcomes in S that are not in E.

E^c is typically called the **complementary event** to E. Moreover,

$$P(E^c) = 1 - P(E).$$

Homework:

1. For the sample space

$$S = \{a, b, c, d\},$$

does the following define a probability distribution?

outcome	probability
a	0.15
b	0.05
c	0.75
d	0.10

Answer: No

2. For the sample space

$$S = \{a, b, c, d, e\},$$

does the following define a probability distribution?

outcome	probability
a	0.15
b	0.05
c	0.75
d	0.02
e	0.03

Answer: Yes

3. For the sample space $S = \{a, b, c, d, e\}$

outcome	probability
a	0.25
b	0.05
c	0.55
d	0.05
e	0.10

Let $E = \{a, c, e\}$. Find $P(E)$ and $P(E^c)$.
Answer: 0.9; 0.1

Tree Diagrams and the Multiplication Principle:

If you are to perform two operations in succession, and the first operation can be performed in N ways and the second operation can be performed in M ways, the the number or ways you can perform the two operations together is $N \cdot M$ ways. This is usually called the **multiplication principle.** The way to visually represent all the outcomes of performing the two successive operations in one after an other is called a **tree diagram**.

Let

$$FO_1, FO_2, FO_3, \cdots, FO_N$$

represent the N possible outcomes of the first operation, and let

$$SO_1, SO_2, SO_3, \cdots, SO_M$$

represent the M possible outcomes of the second operation. Then the following "tree diagram" shows the outcomes of the two operations done in succession, if you follow out all the possible paths or branches from left to right.

Outcome	of 1st op.	Outcome	of 2nd op.
		—	SO_1
—	FO_1	—	SO_2
		⋮	⋮
		—	SO_M
		—	SO_1
—	FO_2	—	SO_2
		⋮	⋮
		—	SO_M
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
		—	SO_1
—	FO_N	—	SO_2
		⋮	⋮
		—	SO_M

Homework: Suppose you roll a "fair" 8 sided die (in the shape of an octahedron), and then you flip a coin. The die has sides labeled 1 through 8, and one side of the coin is H, the other T.

1. What is the sample space for this experiment?
   Answer: 1H, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 1T, 2T, 3T, 4T, 5T, 6T, 7T, 8T

2. What is the probability of rolling an even number?
   Answer: 0.5

3. What is the probability of rolling an H?
   answer: 0.5

4. What is the probability of rolling an even number and an H?
   Answer: 0.25

Example: Suppose that your experiment involved rolling two dice. Each die has six possible outcomes. Thus your sample space has $6 \cdot 6 = 36$ elements in it. *Tree diagram will be drawn in lecture.*

Example: Suppose you are about to select an outfit for a night out. You have laid all the clothes that can be mixed and matched out on your bed. There are 3 pairs of pants, 5 shirts, 6 pairs of socks, and 2 pairs of shoes. How many different outfits are there?

$$3 \cdot 5 \cdot 6 \cdot 2 = 180.$$

OR and AND Problems:

For events A, B if we wish to calculate the probability

$$P(A \quad \text{or} \quad B)$$

we are seeking the probability of the event

$$A \cup B,$$

when we view events as subsets of the sample space S. Likewise if we are seeking the probability

$$P(A \quad \text{and} \quad B)$$

we are seeking the probability of the event

$$A \cap B.$$

Moreover, we actually know the following

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Note that events with no common outcomes are called **mutually exclusive events**. That is A and B are mutually exclusive if and only if

$$A \cap B = \emptyset.$$

Moreover, for mutually exclusive events we have

$$P(A \cap B) = 0$$

and thus

$$P(A \cup B) = P(A) + P(B).$$

Suppose that the following probabilities are known for certain events in an experiment

$$P(A) = 0.3, \quad P(B) = 0.5, \quad P(A \cap B) = 0.15.$$

Find

$$P(A \cup B).$$

Recall that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Thus

$$P(A \cup B) = 0.3 + 0.5 - 0.15 = 0.65.$$

Homework:

Suppose that A and B are events. Suppose that $P(A) = 0.35$ and $P(B^c) = 0.45$ and $P(A \cap B) = 0.27$. Find the following:

1. $P(A^c)$
2. $P(B)$
3. $P(A \cup B)$
4. $P((A \cup B)^c)$
5. $P((A \cap B)^c)$
6. $P(A^c \cap B^c)$
7. $P(A^c \cup B^c)$

Answers: 0.65, 0.55, 0.63, 0.37, 0.73, 0.37, 0.73

An observation that should be noted is the so-called **DeMorgan's Laws:**

$$A^c \cap B^c = (A \cup B)^c,$$

$$A^c \cup B^c = (A \cap B)^c$$

for any two events A, B in a given sample space.

Two events A and B will be called **independent events** if

$$P(A \cap B) = P(A) \cdot P(B).$$

It turns out that A and B are independent events if and only if the occurrence of either event in no way affects the probability of the occurrence of the other event.

To come to a thorough understanding of why this is the case, we must wait until we discuss a topic called **conditional probability**.

Example: Suppose that the probability of having a boy or a girl child is each one half. Find the probability of having three girls.

We assume independence for each birth. Thus, if you were to have a girl in the first pregnancy, this in no way affects whether or not you have a girl in the next pregnancy. Thus, the probability of having three girls over three pregnancies is

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

Conditional Probability:

Earlier we defined **independence** of two events A, B as: A and B are said to be independent if

$$P(A \cap B) = P(A) \cdot P(B).$$

Moreover, it was observed that events A and B are independent if and only if the occurrence of either event in no way effects the probability of the occurrence of the other event.

To see why this is true we have to define something called **conditional probability.** For any two events A, B we define

$$P(A|B)$$

called *"the probability of the event A, given that the event B occurred"* which is sometimes read **the probability of A given B**. We define the conditional probability as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

To see why this is the appropriate definition, we should look at a Venn Diagram. (This will be done in lecture.)

Before we do any specific examples, we should address independence. Events A, B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B).$$

Substituting this in the above formula for $P(A|B)$ we get

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A) \cdot P(B)}{P(B)}$$

$$= P(A).$$

Similarly, we can see that

$$P(B|A) = P(B).$$

Thus, A, B are independent if and only if

$$P(A|B) = P(A)$$

and

$$P(B|A) = P(B).$$

Example: In a certain college, 25 percent of the students failed mathematics, 15 percent failed chemistry and 10 percent failed both subjects. A student is selected at random.

- If the student failed chemistry, what is the probability he/she failed mathematics?

- If the student failed math, what is the probability that he/she failed chemistry?

- What is the probability that the student failed either subject?

- What is the probability that the student failed neither subject?

Let M be the event that the student failed math, and C be the event that the student failed chemistry.

The given info tells us that the probability that a randomly selected student failed math is

$$P(M) = 0.25,$$

the probability that a randomly selected student failed chemistry is

$$P(C) = 0.15$$

and the probability that a randomly selected student failed both subjects is

$$P(M \cap C) = 0.1.$$

Thus, the probability that the student failed math, given that the student failed chemistry is given by

$$P(M|C)$$

which is computed by the following

$$P(M|C) = \frac{P(M \cap C)}{P(C)}$$

$$= \frac{0.1}{0.15} = \frac{2}{3}.$$

The probability that the student failed chemistry, given that the student failed mathematics is given by

$$P(C|M)$$

$$= \frac{P(C \cap M)}{P(M)}$$

$$= \frac{0.1}{0.25} = \frac{2}{5}.$$

Note that we see that C and M are not independent events!

The probability that the student failed either subject is

$$P(M \cup C)$$

and we know that

$$P(M \cup C) = P(M) + P(C) - P(M \cap C)$$

$$0.25 + 0.15 - 0.1 = 0.3.$$

The probability that the students failed neither subject is

$$P(M^c \cap C^c)$$

$$= P((M \cup C)^c)$$

$$= 1 - P(M \cup C)$$

$$1 - 0.3 = 0.7.$$

Example: An actuarial scientist who works for an insurance company determined that a woman has a probability of 89.835 percent chance of living to age 60, while a 57.062 percent chance of living to age 80. Given that a woman is 60, what is the probability that she will live to age 80?

This is a conditional probability question. Let A be the event that the woman lives until age 80, and let B be the event that the woman lives until age 60. We wish to find

$$P(A|B).$$

Now,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A)}{P(B)}$$

$$= \frac{0.57062}{0.89835} \approx 0.63519.$$

Thus, a woman has a 63.519 percent chance of living to age 80 if she has already lived to age 60.

Some Additional Home Work Exercises:

1. Let A and B be events with the following probabilities

   $P(A) = 0.6, \quad P(B) = 0.3, \quad P(A \cap B) = 0.2.$

   Answer the following questions

   (a) Find the conditional probabilities $P(A|B)$ and $P(B|A)$

   (b) Are the events A and B independent?

   (c) Find $P(A \cup B)$.

   (d) Find the probabilities $P(A^c)$ and $P(B^c)$.

   Answers: $0.\overline{6}$ and $0.\overline{3}$, No, 0.7, 0.4 and 0.7

2. Let A and B be two events. Find $P(B|A)$ if

   (a) A is a subset of B.

   (b) B is a subset of A.

   (c) A and B are mutually exclusive (disjoint).

Answers: 1, $\frac{P(B)}{P(A)}$, 0