Math 112: Graphs and Functions

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The Cartesian Coordinate System and Graphs of Equations:

In this section we discuss the **Cartesian coordinate system** or **rectangular coordinate system** for representing the location of points in the plane.

A rectangular coordinate system for the plane consists of two number lines that intersect at the **origin** in the plane at right angles. The horizontal axis is usually referred to as the **x-axis** and the x values increase as you go from left to right. The vertical axis is usually referred to as the **y-axis** and the y values increase as you go up (in the vertical direction or course). In this coordinate system, the origin has coordinates $(0, 0)$.

Points in the plane are identified with orders pairs of numbers (x, y). The first value x is the horizontal coordinate of the point and the second coordinate y is the vertical coordinate of the point.

The x and y-axes cut the plane into four **quadrants.** The **first quadrant** is the region of points (x, y) with $x, y > 0$. The **second quadrant** is the region of points (x, y) with $x < 0$ and $y > 0$. The **third quadrant** is the region of points (x, y) with $x, y < 0$. The **fourth quadrant** is the region of points (x, y) with $x > 0$ and $y < 0$.

In lecture we shall do the following exercises on pages 126-127 of the text:

Numbers 2-10 even. Here we will plot points in the plane given their Cartesian/rectangular coordinates (x, y).

Numbers 18-26 even. Here we will check whether or not a given point (x, y) is a solution to a given equation in the two variables x and y.

We shall also do exercises 28, 34 and 38, which involve graphing the set of solutions (x, y) to a given equation. We will do this by plotting points, which we will do by choosing multiple x values, determining the corresponding y values, plotting these points (x,y) and then "connecting the dots".

Homework Exercises from pages 126-127:
1-9 odd, 17-25 odd, 27-33 odd, 37, 47.

An Introduction to Functions of One Variable:

A **function** is a relation between two collections of objects or **sets**. *

Given two sets A, B, a **relation** is a collection of ordered pairs (x, y), where x is in A $(x \in A)$ and y is in B $(y \in B)$.

Functions are relations for which each first component in the ordered pairs corresponds to exactly one second component.

*In this course we shall deal only with functions which are relations between sets of numbers.

The **domain** of a function is the set of values in the set A that appear in the first coordinate in the collection of ordered pairs. Commonly A is the domain. The **range** of the function is the collection of second coordinates in the collection of ordered pairs. The set B is usually referred to as the **codomain.** Note that the range is a subset of the codomain.

We usually refer to a function by the formula that gives the relation between the two coordinates x and y in the collection of ordered pairs. Here that will be given by a formula of the form

$$y = f(x)$$

where x is a value in the domain, and y is the corresponding second coordinate of (x, y) in the relation. Moreover, $f(x)$ is the formula for how to compute y in terms of x.

Saying that a relation is a function is simply saying that for each x, there is a unique y if there is any ordered pair with coordinates (x, y) given by the relation. That is x cannot be paired with two different $y's$.

If given the function $y = f(x)$ without a specific domain specified, we shall assume the domain is the so-called **natural domain.** This is the collection of all possible values for which the formula $y = f(x)$ makes sense. We note that when discussing a function, it is important that we know both the formula for y in terms of x – that is $y = f(x)$ – as well as the domain of the function.

The **graph** of a function is the collection of points $(x, f(x))$ plotted in the plane, where x varies over the domain of the function.

A curve represents the graph of a function if it passes the so-called **vertical line test:**

If no vertical line can be drawn so that it intersects the curve more than once, the the curve is the graph of a function.

In lecture we shall do some of the following exercises on pages 141-141 numbers:

2,4, 12 – Determining whether a relation is a function.

24, 26, 28 – Determining if a given curve is the graph of a function.

44-54 even - Determining if a formula relating x and y determines a function.

56-70 even – Evaluating functions.

Homework on pages 141-143 numbers: 1, 3, 5, 7, 11, 23, 25, 27, 43-53 odd, 55-69 odd

Linear Functions and Graphing Lines:

A **linear function** is a function that can be written in the form

$$y = mx + b$$

where m, b are constants (fixed real numbers). The domain of a linear function is $\mathbb{R}$, the set of real numbers.

Since two points uniquely determine a line, we only need to find two points, and then connect the dots, to graph a line.

The $x - intercept$ of the graph of a line is the x-coordinate when the line crosses the x-axis. this is given by solving

$$0 = mx + b$$

for x. the $y{-}intercept$ is the y-coordinate where the line crosses the y-axis. It is given by computing

$$f(0).$$

In lecture we shall do some of the following exercises: page 151 numbers 2, 4, 6, 8, 14, 18, 20, 24, 26, 28, 36, 38, 40

Homework exercises: page 151 numbers 1-7 odd, 9-12, 13-19 odd, 23-27 odd, 29-32, 35-39 odd

The **slope** of a line is a quantity that measures how steep a line is. Given any two points

$$(x_1, y_1), \quad (x_2, y_2)$$

on the line, the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example: Find the slope of the line that passes through the points

$$(0, 3), \quad (2, 5).$$

$$\begin{aligned} m &= \frac{5 - 3}{2 - 0} \\ &= 1. \end{aligned}$$

In lecture we shall also do some of the following exercises: page 163 numbers 2, 4, 8, 12, 14.

For Homework do the following exercises: page 163 numbers 1-17 odd.

There is a connection between the slope of a line and how the line rises or falls:

- The slope is positive if and only if the line rises as you go from left to right.

- The slope is negative if and only if the line falls as you go from left to right.

- A line has slope 0 if and only if the line is a horizontal line.

- A line is a vertical line if and only if its slope is undefined.

The **slope-intercept** form of a line is

$$y = mx + b$$

where here m is the slope of the line, and b is the y-intercept of the line.

Example: Find the slope intercept of the line $4x - 8y = 12$.

Here we solve for y, to get:

$$4x - 8y = 12$$

$$-8y = -4x + 12$$

$$y = \frac{-4}{-8}x + \frac{12}{-8}$$

$$y = \frac{1}{2}x - \frac{3}{2}$$

In lecture we shall also do some of the following exercises: page 163 numbers 20-32 even.

For Homework do the following exercises: page 163 numbers 19-31 odd.

Given two lines L_1 with slope m_1 and L_2 with slope m_2 then:

- L_1 and L_2 are parallel if and only if $m_1 = m_2$

- L_1 and L_2 are perpendicular if and only if $m_1 \cdot m_2 = -1$

In lecture we shall also do some of the following exercises: page 164 numbers 60-64 even.

For Homework do the following exercises: page 164 numbers 59-65 odd.

Another important form of a line is the so-called **point-intercept form** of the line. Given a point on the line (x_1, y_1) and the slope of the line m, a point-intercept form is given by

$$y - y_1 = m(x - x_1).$$

Example: Find a slope-intercept and the point-slope forms of the line through the points

$$(-1, 2) \quad and \quad (3, 5).$$

First we find the slope:

$$m = \frac{5-2}{3-(-1)} = \frac{3}{4}.$$

Thus a slope-intercept form is

$$y - 5 = \frac{3}{4}(x - 3).^*$$

Solving this for y, we will obtain the point-slope form of the line

$$y - 5 = \frac{3}{4}x - \frac{9}{4}$$

$$y = \frac{3}{4}x + \frac{11}{4}.$$

*or equivalently $y - 2 = \frac{3}{4}(x + 1)$.

In lecture we shall also do some of the following exercises: pages 173-174 numbers 16, 22, 26, 30, 42-50 even.

For Homework do the following exercises: pages 173-174 numbers 13- 29 odd, 41-51 odd. odd.

Earlier we discussed the so-called point-slope form a line

$$y - y_1 = m(x - x_1),$$

where (x_1, y_1) is a point on the line and m is the slope of the line. Also, we discussed the slope-intercept form of a line

$$y = mx + b,$$

where b is the y-intercept of the line and m is the slope of the line.

There is one more form of a line which we should discuss, which is the so-called **general form** of the line

$$Ax + By = C.$$

Here A, B, C are constants – that is fixed real numbers.

Example: Consider the line

$$3x + 4y = 5.$$

Find the x and y-intercepts of this line.

Setting $y = 0$ gives us the x-intercept. That is

$$3x = 5$$

or simply

$$x = \frac{5}{3}.$$

Setting $x = 0$ gives us the y-intercept. That is

$$4y = 5$$

or simply

$$y = \frac{5}{4}.$$

Graphing Linear Inequalities:

In this section we shall consider inequalities of the form

- $$Ax + By \leq C$$

- $$Ax + By < C$$

- $$Ax + By \geq C$$

- $$Ax + By > C$$

where A, B, C are constants.

To solve any of these inequalities, we will first graph the line

$$Ax + By = C.$$

Then we will observe that the solution to the inequality in question is a half-plane whose boundary is the line $Ax + By = C$. In the case where there is a non-strict inequality, the line itself is part of the solution region. In the case of a strict inequality, the line is not part of the solution region.

To determine which half plane is the solution of the inequality, we simply test a point clearly on one side of the line. If the inequality is satisfied at this point, then that half-plane is the solution region. Otherwise it is the other half-plane.

Example: Solve the inequality

$$2x - y < 6.$$

Here, we first graph the line

$$2x - y = 6.$$

This line has x-intercept $x = 3$ and y-intercept $y = -6$.

We then test the point $(0, 0)$, which is on the left half-plane (see graph). Clearly

$$2 \cdot 0 - 0 = 0 < 6.$$

Thus the half-plane that contains the point $(0, 0)$ – not including the line itself – is the solution set of the inequality.

In lecture we shall also do some of the following exercises: page 189 numbers 2, 6, 8, 10, 12.

For Homework do the following exercises: page 189 numbers 1, 5, 7, 9, 11.

Next we shall find the solution to combinations of linear inequalities with **and**'s or **or**'s.

To solve such inequalities linked with an and or an or, we first solve each inequality separately. Then if there is an **and**, we take the intersection of the two solution regions – that is, what is in common. If there is an **or**, we take the union of the two solution regions – that is, what is in either region.

In lecture we shall also do some of the following exercises: page 189 numbers 16-22 even.

For Homework do the following exercises: page 189 numbers 15-21 odd.

Solving Systems of Linear Equations:

In this section we are going to solve two linear equations simultaneously

$$Ax + By = C,$$

$$Dx + Ey = F.$$

Solutions are points that line on both lines. Thus we see that there are three possibilities:

- All points on each line are solutions – that is the two lines are really the same line.
- There is one distinct point in common, namely where the two lines intersect.
- There are no points in common – that is, the lines are parallel.

We will consider two methods for solving the system in question, namely the **substitution method** and the **elimination method.**

The substitution method:

- Solve one equation for one of its variables.
- Substitute this in the other equation.
- Find the solution(s) to the equation obtained after making the substitution.
- After obtaining this (these) solution(s), substitute this (these) value(s) in one of the two equations, and solve for the value(s) of the other variable.
- Check your answer(s).

The elimination method:

- Rewrite each equation in the general form if they are not already in this form.

- If necessary, multiply one or both of the equations by some non-zero constant(s) so that the coefficients in one of the two variables are negatives of one another.

- Add the two equations together.

- Find the value(s) of the remaining variable in this new equation you obtained.

- Substitute this (these) values in one of the original equations, and solve for the value(s) of the other variable.

- Check your answer(s).

In lecture we shall also do some of the following exercises: page 213 numbers 8, 10, 12, 20, 24, 32, 34.

For Homework do the following exercises: page 213 numbers 7, 9, 11, 17, 21, 23, 31, 33.