Math 112: Rational Expressions

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Summer 2008

Multiplying and Dividing Rational Expressions :

In this section we shall study what are called **rational functions**, which are functions of the form

$$f(x) = \frac{a_m x^m + \cdots + a_2 x^2 + a_1 x + a_0}{b_n x^n + \cdots + b_2 x^2 + b_1 x + b_0},$$

where m, n are non-negative integers, and

$$a_0, \cdots, a_m, b_0, \cdots, b_n \in \mathbb{R}.$$

The domain of this function is

$$\{x : b_n x^n + \cdots b_1 x + b_0 \neq 0\}.$$

A **rational expression** is an expression of the form

$$\frac{P}{Q}$$

where P, Q are polynomials in some variable x.

Rational functions are functions of some variable x whose formula for computing the function value are given by a rational expression.

Simplifying Rational Expressions:

For any rational expression $\frac{P}{Q}$ and any polynomial R, where $R \neq 0$, we have

$$\frac{PR}{QR} = \frac{P}{Q}.$$

For example:

$$\frac{(x+2)^2}{x^2-4} = \frac{(x+2)(x+2)}{(x-2)(x+2)} = \frac{x+2}{x-2}$$

provided $x \neq \pm 2$.

Multiplying and Dividing Rational Expressions:

For any two rational expressions $\frac{P}{Q}, \frac{R}{S}$ with $Q, S \neq 0$ we have multiplication defined by

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{P \cdot R}{Q \cdot S}.$$

Likewise, we may define division by

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{P \cdot S}{Q \cdot R}$$

Provided $R \neq 0$ as well.

Examples:

$$\frac{x^3 - 1}{-3x + 3} \cdot \frac{15x^2}{x^2 + x + 1}$$

$$= \frac{(x-1)(x^2 + x + 1)}{-3(x-1)} \cdot \frac{15x^2}{x^2 + x + 1}$$

$$= \frac{(x-1)(x^2 + x + 1)15x^2}{-3(x-1)(x^2 + x + 1)}$$

$$= \frac{5x^2}{-1} = -5x^2.$$

$$\frac{18y^2 + 9y - 2}{24y^2 - 10y + 1} \div \frac{3y^2 + 17y + 10}{8y^2 + 18y - 5}$$

$$= \frac{18y^2 + 9y - 2}{24y^2 - 10y + 1} \cdot \frac{8y^2 + 18y - 5}{3y^2 + 17y + 10}$$

$$\frac{(6y - 1)(3y + 2)}{(6y - 1)(4y - 1)} \cdot \frac{(4y - 1)(2y + 5)}{(3y + 2)(y + 5)}$$

$$= \frac{2y + 5}{y + 5}.$$

In lecture we shall do some of the following exercises: pages 348-349 numbers 2, 6, 10, 16, 18, 20, 28, 30, 34, 38, 40, 46, 50, 54

Homework: pages 348-349 numbers 1, 5, 9, 13-33 odd, 37, 39, 43, 49, 51, 53

Adding and Subtracting Rational Expressions:

If $\frac{P}{Q}$ and $\frac{R}{Q}$ are rational expressions, $Q \neq 0$, then we may define addition and subtraction by

$$\frac{P}{Q} + \frac{R}{Q} = \frac{P + R}{Q}$$

and

$$\frac{P}{Q} - \frac{R}{Q} = \frac{P - R}{Q}.$$

Examples:

$$\frac{x^2}{x+7} - \frac{49}{x+7} = \frac{x^2-49}{x+7}$$

$$= \frac{(x+7)(x-7)}{x+7}$$

$$= x-7$$

$$\frac{x}{x^2-1} + \frac{1}{x^2-1}$$

$$= \frac{x+1}{x^2-1}$$

$$= \frac{x+1}{(x+1)(x-1)}$$

$$= \frac{1}{x-1}.$$

If you wish to add two rational expressions or subtract one rational expression from another whose denominators aren't the same, then you may do so by finding a common denominator. One such way is

$$\frac{P}{Q} + \frac{R}{S} = \frac{PS + QR}{QS}$$

and

$$\frac{P}{Q} - \frac{R}{S} = \frac{PS - QR}{QS}.$$

In lecture we shall do some of the following exercises: page 357 numbers 4, 6, 8, 10, 24-30 even; page 364 numbers 4, 6, 8, 14

Homework: page 357 numbers 3-9 odd, 23-29 odd; page 364 1-7 odd, 11, 13, 15

Polynomial Division:

In this section we shall discuss how to perform long division of polynomials. Given two polynomials P, Q we would like to compute

$$\frac{P}{Q},$$

whenever the degree of P is greater than or equal to the degree of Q. In such a case, we write

$$\frac{P}{Q} = S + \frac{R}{Q}$$

where S, R are polynomials, and the degree of R is less than the degree of Q.

In lecture we shall do some of the following exercises: pages 373-374 numbers 8-14 even, 18, 20, 38

Homework: pages 373-374 numbers 7-19 odd, 37

Solving Equations with Rational Expressions:

In this section we will consider equations which involve rational expressions in some variable x. We wish to find the solutions to such equations. The principle that we shall follow is as follows

- Bring all terms to one side of the equation.
- The side which is not zero, write as a single fraction.
- Observe that a fraction equals zero if and only if its numerator is zero. The places where the denominator is zero merely give restrictions on our variable x.

- Set the denominator equal to zero to find restrictions on x.

- Set the numerator equal to zero to solve the problem.

Example: Solve

$$\frac{2}{x^2-4} = \frac{1}{2x-4}$$

Solution:

$$\frac{2}{x^2-4} = \frac{1}{2x-4}$$

$$\frac{2}{x^2-4} - \frac{1}{2x-4} = 0$$

$$\frac{2}{(x-2)(x+2)} - \frac{1}{2(x-2)} = 0$$

$$\frac{2}{2} \cdot \frac{2}{(x-2)(x+2)} - \frac{1}{2(x-2)} \cdot \frac{x+2}{x+2} = 0$$

$$\frac{4}{2(x-2)(x+2)} - \frac{x+2}{2(x-2)(x+2)} = 0$$

$$\frac{4-(x+2)}{2(x-2)(x+2)} = 0$$

$$\frac{2-x}{2(x-2)(x+2)} = 0$$

The restrictions on x are given by solving

$$2(x-2)(x+2)=0$$

which is $x=2,-2$. Thus the restrictions are

$$x\neq 2,-2.$$

The solutions are given by solving

$$2-x=0$$

which is only $x=2$. However, we note that $x\neq 2$ is a restriction. Thus there are no solutions to the original problem.

In lecture we shall do some of the following exercises: pages 380-381 numbers 6-16 even, 22

Homework: pages 380-381 numbers 3-21 odd