

Math 243: Parameterized
Curves, and Polar Coordinates
for \mathbb{R}^2

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Parameterized Curves:

Let \mathcal{C} be a curve in \mathbb{R}^2 – or more generally in \mathbb{R}^n . A curve \mathcal{C} is **parameterized** if its coordinates can be written as functions of a single variable, say t , with a common domain I , an interval in \mathbb{R} .

That is, for curves in \mathbb{R}^2 we have

$$\mathcal{C} = \{(x(t), y(t)) : x, y : I \rightarrow \mathbb{R}\}.$$

That is, for curves in \mathbb{R}^3 we have

$$\mathcal{C} = \{(x(t), y(t), z(t)) : x, y, z : I \rightarrow \mathbb{R}\}.$$

That is, for curves in \mathbb{R}^n we have

$$\mathcal{C} = \{(x_1(t), x_2(t), \dots, x_n(t)) : x_1, x_2, \dots, x_n : I \rightarrow \mathbb{R}\}.$$

Some Examples:

- Let $f : I \rightarrow \mathbb{R}$, then the graph of f

$$\mathcal{G} = \{(t, f(t)) : t \in I\}$$

is a parameterized curve.

- Let \mathcal{C} be the unit circle $\{(x, y) : x^2 + y^2 = 1\}$. Then \mathcal{C} may be parameterized by

$$\mathcal{C} = \{(\cos \theta, \sin \theta) : \theta \in [0, 2\pi]\}.$$

- The spiral in \mathbb{R}^3 given by

$$\{(\cos t, \sin t, t) : t \in \mathbb{R}\}$$

is a parameterized curve.

- The ellipse

$$\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$$

may be parameterized as follows:

$$\{(a \cos t, b \sin t) : t \in [0, 2\pi]\}.$$

- The curve

$$\{(x, y) : x^4 + y^4 = 1\}$$

may be parameterized by

$$\{(f(t), g(t)) : t \in [0, 2\pi]\}$$

where

$$f(t) = \text{sign}(\cos t) \sqrt{|\cos t|}$$

and

$$g(t) = \text{sign}(\sin t) \sqrt{|\sin t|}$$

where $\text{sign}(x) = 1$ if $x \geq 0$ and -1 if $x < 0$.

If a parameterized curve \mathcal{C} in \mathbb{R}^2 is given by

$$x = f(t), \quad y = g(t), \quad t \in [a, b],$$

then we call $(f(a), g(a))$ the **initial point** and we call $(f(b), g(b))$ the **terminal point**.

We call a parameterized curve **closed** if

$$(f(a), g(a)) = (f(b), g(b)).$$

We call a parameterized curve **simple** if

$$(f(t), g(t)) \neq (f(\tau), g(\tau))$$

for any $t \neq \tau$ with $t, \tau \in (a, b)$. In other words, there are no self intersections.

Tangents to Parameterized Curves:

Here we restrict ourselves to \mathbb{R}^2 , however in \mathbb{R}^n the results are similar.

Suppose

$$\mathcal{C} = \{(x(t), y(t)) : t \in [a, b]\}.$$

If $x(t), y(t)$ are differentiable functions of t , then the curve given by

$$x = x(c) + tx'(c), \quad y = y(c) + ty'(c), t \in \mathbb{R}$$

is the tangent line to \mathcal{C} at the point $(x(c), y(c))$ on \mathcal{C} .

For example, for the parameterized curve

$$x = \cos t, \quad y = \sin t, \quad t \in [0, 2\pi]$$

we have that at $t = \frac{\pi}{3}$ the tangent line is given by

$$x = x\left(\frac{\pi}{3}\right) + tx'\left(\frac{\pi}{3}\right), \quad y = y\left(\frac{\pi}{3}\right) + ty'\left(\frac{\pi}{3}\right), \quad t \in \mathbb{R}$$

that is

$$x = \cos \frac{\pi}{3} - t \sin \frac{\pi}{3}, \quad y = \sin \frac{\pi}{3} + t \cos \frac{\pi}{3}, \quad t \in \mathbb{R},$$

or simply

$$x = \frac{1}{2} - t \frac{\sqrt{3}}{2}, \quad y = \frac{\sqrt{3}}{2} + t \frac{1}{2}, \quad t \in \mathbb{R}.$$

Moreover, if y may be locally written as a function of x , by the chain rule we see that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

and thus if $\frac{dx}{dt} \neq 0$ we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ for the parameterized curve given by

$$x = 2 \cos t, \quad y = 3 \sin t.$$

Here

$$\frac{dy}{dt} = \frac{-2 \sin t}{3 \cos t}$$

and thus at $t = \frac{\pi}{4}$ we have

$$\frac{dy}{dx} \Big|_{(\sqrt{2}, \frac{3}{\sqrt{2}})} = -\frac{2}{3}.$$

Area Under Parameterized Curves:

If $y = F(x)$, $F > 0$ is a function, the the area under the graph of F , above the x -axis and between $x = a$ and $x = b$ is given by

$$A = \int_a^b F(x)dx.$$

If a curve is given parametrically by $x = f(t)$ and $y = g(t) > 0$ for $t \in [\alpha, \beta]$ with x increasing as t increases, then the area between the curve, the x -axis and the lines $x = f(\alpha)$ and $x = f(\beta)$ is given by

$$A = \int_{\alpha}^{\beta} g(t)f'(t)dt.$$

Example: Find the area of the region bounded by the curve $x^2 + y^2 = 1$.

To solve this we will find half of this area, and use symmetry to find the total area. Here we examine the area under the parameterized curve given by $x = \cos(\pi - t)$, $y = \sin(\pi - t)$ for $t \in [0, \pi]$ and above the x -axis.

This is given by

$$\begin{aligned} A &= \int_0^{\pi} \sin^2(\pi - t) dt \\ &= -\frac{1}{2}(\pi - t) + \frac{1}{4} \sin 2(\pi - t) \Big|_{t=0}^{\pi} \\ &= \frac{\pi}{2}. \end{aligned}$$

Thus the total area is π , as expected.

Arc Length of Parameterized Curves:

Suppose that \mathcal{C} is a curve given parametrically by

$$x = x(t), \quad y = y(t), \quad t \in [a, b].$$

Suppose that $x(t)$ and $y(t)$ are differentiable for $t \in [a, b]$ and their first derivatives $x'(t), y'(t)$ are continuous. We wish to find the **length** of \mathcal{C} , often called the **arclength** of the curve \mathcal{C} .

First we shall restrict ourselves to the special case of the graph of a function $y = f(x)$.

The **length** L of the curve given by the graph of $y = f(x)$ over $[a, b]$ shall be defined as follows: For simplicity, let us assume that $f(x)$ is differentiable on $[a, b]$, with a continuous first derivative on $[a, b]$. Otherwise the length may not be defined.*

*For example, consider $y = f(x)$ on $[0, 1]$ given by $f(0) = 0$ and $f(x) = x \sin(\frac{1}{x})$ for $0 < x \leq 1$. This curve would have to have infinite length.

We will assume that the length of a line segment is the only length thus known, and this is simply the value you obtain using the distance formula with the endpoints of the line segment.

So consider a partition of the interval $[a, b]$

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

and define

$$\Delta x_i := x_i - x_{i-1}.$$

For $1 \leq i \leq n$, consider the graph of $y = f(x)$ on $[x_{i-1}, x_i]$. Assuming that Δx_i is sufficiently small, by the differentiability of $f(x)$, we have that the length of the graph of $f(x)$ on $[x_{i-1}, x_i]$ is approximately

$$l_i := \sqrt{\Delta x_i^2 + (f(x_i) - f(x_{i-1}))^2}.$$

This is simply the length of the line segment with endpoints $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$.

However, we may rewrite l_i as

$$l_i = \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{\Delta x_i}\right)^2} \Delta x_i.$$

Moreover, using the mean value theorem, we see that there is a number $\xi_i \in (x_{i-1}, x_i)$ so that

$$l_i = \sqrt{1 + (f'(\xi_i))^2} \Delta x_i.$$

Thus, L may be approximated by

$$\sum_{i=1}^n \sqrt{1 + (f'(\xi_i))^2} \Delta x_i$$

which is a Riemann sum for the integral

$$\int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Moreover, by our assumptions on f , the larger n gets in the above approximation for L , the smaller

$$\left| L - \sum_{i=1}^n \sqrt{1 + (f'(\xi_i))^2} \Delta x_i \right|$$

gets. Thus

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Now, consider a parameterized curve in \mathbb{R}^2 :

$$\vec{r}(t) = (x(t), y(t)),$$

$t \in [a, b]$ and $x(t), y(t)$ differentiable with continuous first derivatives on $[a, b]$.*

For parameterized curves $(x(t), y(t))$, we wish to determine the **length** L of the curve over the interval $[a, b]$.

*If $x(t) = t$, then this case reduces to the graph of a function $y = f(t)$, which we just discussed.

Once again, we partition $[a, b]$ by

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b,$$

and define $\Delta t_i = t_i - t_{i-1}$.

Let l_i be the length of the piece of the curve for t satisfying $[t_{i-1}, t_i]$. Then we may approximate l_i by

$$\sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$$

which by the mean value theorem, may be written as

$$\sqrt{(x'(\xi_i))^2 + (y'(\eta_i))^2} \Delta t_i$$

for some $\xi_i, \eta_i \in (t_{i-1}, t_i)$.

Hence, we see that

$$L \approx \sum_{i=1}^n \sqrt{(x'(\xi_i))^2 + (y'(\eta_i))^2} \Delta t_i$$

for n sufficiently large.

Letting $n \rightarrow \infty$ we get

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

Example:

Find the length of the curve given parametrically by

$$x(\theta) = \cos \theta, \quad y(\theta) = \sin \theta \quad \theta \in [0, 2\pi].$$

Here we have

$$x'(\theta) = -\sin \theta, \quad y'(\theta) = \cos \theta$$

and thus

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-\sin \theta)^2 + (\cos \theta)^2} d\theta \\ &= \int_0^{2\pi} d\theta = 2\pi. \end{aligned}$$

Polar Coordinates:

Given a point $(x, y) \in \mathbb{R}^2$, x, y not both 0, we may express x and y in the following form:

$$x = r \cos \theta, \quad y = r \sin \theta$$

where

$$r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}.$$

Thus $r \geq 0$ and $\theta \in [0, 2\pi)$. This representation of x and y in terms of (r, θ) , called the **polar coordinates** of (x, y) , is a representation where θ is the angle between the ray determined by $(0, 0)$ and (x, y) and the positive x -axis, and r is the Euclidean distance between (x, y) and $(0, 0)$.

For example, the circle \mathcal{C} given by

$$x^2 + y^2 = 4$$

is given by

$$r = 2$$

in polar coordinates.

The curve given by

$$x^2 - y^2 = 1$$

is given by

$$r^2(\cos^2 - \sin^2) = 1.$$

Areas and Lengths in Polar Coordinates:

If a curve \mathcal{C} is given in polar coordinates with r as a function of θ ,

$$r = r(\theta),$$

we have that the area of the region bounded by $\theta = a$, $\theta = b$ and $r = r(\theta)$ is given by

$$A = \int_a^b \frac{1}{2} r^2 d\theta.$$

The length of \mathcal{C} for $\theta \in [a, b]$ is given by

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$