Math 243: Real-Valued Functions of Several Variables and Integration

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Fall 2009

Integrals of Functions of Two Variables over Rectangles:

Let

$$z = f(x, y)$$

be a function of two variables defined on a rectangle $R = [a, b] \times [c, d]$. Suppose we partition $[a, b]$ in the following manner

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and we partition $[c, d]$ by

$$c = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = d.$$

These partitions give rise to a partition $\mathcal{P}$ of R into smaller rectangles by considering all intervals of the form

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

where

$$1 \leq i \leq n, \quad and \quad 1 \leq j \leq m.$$

Choosing a representative point $(x_{ij}*, y_{ij}*)$ from each R_{ij}, we consider the **Riemann Sum** of $f(x, y)$ on $R = [a, b] \times [c, d]$ with respect to the above defined partition as

$$\sum_{1 \leq i \leq n, 1 \leq j \leq m} f(x_{ij}*, y_{ij}*) \triangle A_{ij}$$

where $\triangle A_{ij}$ is the area of R_{ij}. That is, if $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, then $\triangle A_{ij} = (x_i - x_{i-1}) \cdot (y_j - y_{j-1})$.

We define the **norm of the partition** $\mathcal{P}$ to be the maximum length of a diagonal of all the rectangle R_{ij} in the partition $\mathcal{P}$ of R; this we call write as

$$\|\mathcal{P}\|.$$

That is,

$$\|\mathcal{P}\| = \max_{1\leq i\leq n, 1\leq j\leq m} \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}.$$

We define the **Riemann Integral** of f on R by

$$\int\int_R f(x,y)dA =$$

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{1\le i\le n, 1\le j\le m} f(x_{ij}*, y_{ij}*)\triangle A_{ij}$$

provided the limit exists. Here we note that the limit is taken over all all possible partitions $\mathcal{P}$ of R with any choice of representative points $(x_{ij}*, y_{ij}*)$ taken from each A_{ij} in any fixed partition $\mathcal{P}$ of R.

The Existence of $\int\int_R f(x,y)dA$ when f is continuous:

It can be show that if $z = f(x,y)$ is continuous, then

$$\int\int_R f(x,y)dA$$

as defined above, exists.

The Geometric Meaning of the Riemann Integral:

Suppose that $z = f(x, y)$ is a non-negative function, then

$$\int\int_R f(x, y)dA$$

is the volume of the region bounded above by the graph of $z = f(x, y)$ and below by R is the xy-plane.

Iterated Integrals and Fubini's Theorem:

If

$$z = f(x, y)$$

is a continuous function defined on $R = [a, b] \times [c, d]$ then the integral

$$\int\int_R f(x, y)dA$$

equals the following:

$$\int\int_R f(x, y)dA = \int_a^b (\int_c^d f(x, y)dy)dx$$

and

$$\int\int_R f(x, y)dA = \int_c^d (\int_a^b f(x, y)dx)dy.$$

Example: Evaluate the integral

$$\int\int_{[1,2]\times[0,1]} x^2 y^5 dA$$

Solution: by Fubini's theorem we have

$$\int\int_{[1,2]\times[0,1]} x^2 y^5 dA = \int_1^2 (\int_0^1 x^2 y^5 dy) dx$$

$$= \int_1^2 (\frac{1}{6} x^2 y^6 |_{y=0}^{y=1}) dx$$

$$= \int_1^2 \frac{1}{6} x^2 dx$$

$$= \frac{1}{18} x^3 |_{x=1}^{x=2}$$

$$= \frac{1}{18}(8-1) = \frac{7}{18}.$$

Example: Evaluate the integral

$$\int\int_{[0,\frac{\pi}{2}]\times[0,1]} x \cos xy dA.$$

Solution: We use Fubini's theorem, and obtain the value of the integral through a computation of the iterated integral

$$\int_0^1 \int_0^{\frac{\pi}{2}} (\int_0^1 x \cos xy dy) sx$$

$$= \int_0^{\frac{\pi}{2}} \frac{x \sin xy}{x} |_{y=0}^{y=1} dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x dx$$

$$= -\cos x|_{x=0}^{x=\frac{\pi}{2}}$$

$$= 1.$$

Integrals over Non-rectangular Regions:

Here, we consider

$$\int\int_D f(x,y)dA$$

where D is a (not necessarily rectangular) region contained in the domain of a continuous function

$$z = f(x,y).$$

Let R be a rectangle that contains D. By extending $z = f(x,y)$ in the following matter

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

we may define

$$\int\int_D f(x,y)dA = \int\int_R F(x,y)dA.$$

If $z = f(x, y)$ is continuous on D then this integral will exist. Moreover, if D is a region of **type 1** or **type 2**, then we may use Fubini's theorem to evaluate the integral as an iterated integral.

Here a type 1 region is a region of the form

$$D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

A type 2 region is a region of the form

$$D = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}.$$

If D is a type 1 region, then

$$\int\int_D f(x,y)dA = \int_a^b (\int_{g_1(x)}^{g_2(x)} f(x,y)dy)dx$$

and if D is a type 2 region, then

$$\int\int_D f(x,y)dA = \int_c^d (\int_{h_1(y)}^{h_2(y)} f(x,y)dx)dy$$

provided g_1, g_2, h_1, h_2 are nice enough (say differentiable) and $f(x,y)$ is continuous on D.

Example: Evaluate

$$\int\int_D \frac{4y}{x^3+2} dA$$

where

$$D = \{(x,y) : 1 \leq x \leq 2, 0 \leq y \leq 2x\}.$$

Solution:

We use Fubini's theorem to write this integral as

$$\int_1^2 (\int_0^{2x} \frac{4y}{x^3+2} dy) dx = \int_1^2 \frac{2y^2}{x^3+2}|_{y=0}^{y=2x} dx$$

$$= \int_1^2 \frac{8x^2}{x^3+2} dx$$

$$=_{u=x^3+2} \frac{8}{3} \int_3^{10} \frac{1}{u} du$$

$$= \frac{8}{3} \ln u|_{u=3}^{u=10} = \frac{8}{3}(\ln 10 - \ln 3).$$

Example: Use Fubini's Theorem to rewrite the iterated integral by reversing the order of integration, and evaluate it:

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}dxdy.$$

Solution:

Here we write

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}dxdy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}dydx$$

$$= \int_0^1 \sqrt{x^3+1}y|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 \sqrt{x^3+1} \cdot x^2 dx$$

$$=_{u=x^3+1} \frac{1}{3}\int_1^2 \sqrt{u}du$$

$$= \frac{2}{9}u^{\frac{3}{2}}|_{u=1}^{u=2} = \frac{2}{3}(2^{\frac{3}{2}}-1).$$