Math 311: The Algebra of Matrices

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Matrices:

An $m \times n$ **matrix** A over $\mathbb{R}$ * is an array of real numbers[†] of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Such a matrix A has m rows and n columns.

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[*] or equivalently, over $\mathbb{C}$, or any field $\mathbb{F}$

[†] or complex numbers, or elements of the background field $\mathbb{F}$

The ith row of the matrix A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix},$$

for $1 \leq i \leq m$.

The jth column of the matrix A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix},$$

for $1 \leq j \leq n$.

The element a_{ij}, is the entry in the ith row and the jth column of A.

Often we write

$$[a_{ij}]_{1\leq i\leq m, 1\leq j\leq n}$$

or simply $[a_{ij}]$ to represent the $m \times n$ matrix A.

An $m \times n$ matrix A and an $k \times l$ matrix B are said to be equal, if and only if

$$m = k, \quad n = l$$

and

$$a_{ij} = b_{ij}$$

for all entries $1 \le i \le m$ and $1 \le j \le n$.

We shall use the notation

$$0_{m,n}$$

to represent the $m \times n$ matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This matrix is often called the $m \times n$ **zero matrix**.

Examples:

1. The matrix
$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$
is a 2×3 matrix.

2. The matrix
$$0_{4,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
is the 4×2 zero matrix.

Algebra on $m \times n$ Matrices:

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. The **sum** of A and B, denoted by

$$A + B,$$

is the $m \times n$ matrix whose ij-entry is $a_{ij} + b_{ij}$, that is

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}.$$

Given a scalar k, and an $m \times n$ matrix $A = [a_{ij}]$, the **scalar product** of A by k, denoted

$$kA,$$

is the $m \times n$ matrix whose ij-entry is ka_{ij}, that is

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}.$$

Given an $m \times n$ matrix $A = [a_{ij}]$, we define the **negative** of A, denoted by

$$-A,$$

to be the matrix $(-1)A$, that is the ijentry of $-A$ is $-a_{ij}$.

Given two $m \times n$ matrices A, B, we define the **difference**

$$A - B$$

to be the matrix $A + (-B)$.

We shall observe the following order of operations:

Given two $m \times n$ matrices A, B and two scalars c, k, we shall use

$$kA + cB$$

to denote the quantity

$$(kA) + (cB).$$

Moreover, we call such a quantity a **linear combination** of A and B.

Examples: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 4 & 6 \end{bmatrix}.$$

1.

$$-A = \begin{bmatrix} -1 & -2 & -3 \\ 4 & -2 & 1 \end{bmatrix}$$

2.

$$A + B = \begin{bmatrix} 1 & 3 & 3 \\ -1 & 6 & 5 \end{bmatrix}$$

3.

$$2B = \begin{bmatrix} 0 & 2 & 0 \\ 6 & 8 & 12 \end{bmatrix}$$

4.

$$A - B = \begin{bmatrix} 1 & 1 & 3 \\ -7 & -2 & -7 \end{bmatrix}$$

5.

$$3A + 2B = \begin{bmatrix} 3 & 8 & 9 \\ -6 & 14 & 9 \end{bmatrix}$$

Algebraic Properties of Matrix addition and Scalar Multiplication:

Let A, B, C be $m \times n$ matrices and k, c scalars, then we have the following:

1. Associativity of Addition

$$(A + B) + C = A + (B + C)$$

2. Addition by 0

$$A + 0_{m,n} = 0_{m,n} + A = A$$

3. Existence of an Additive Identity

$$A + (-A) = -A + A = 0_{m,n}$$

4. Commutativity of Addition

$$A + B = B + A$$

5. Multiplication by 1

$$1A = A$$

6. Associativity of Scalar Multiplication

$$(kc)A = k(cA)$$

7. The Distributive Property

$$k(A + B) = (kA) + (kB)$$

8. The Distributive Property

$$(k + c)A = (kA) + (cA)$$

Matrix Multiplication:

Let A be an $m \times n$ matrix and B an $n \times p$ matrix. That is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}.$$

We define the **product** of the two matrices A and B, denoted by

$$AB,$$

to be the matrix C whose ij-entry c_{ij} is given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

That is, the element c_{ij} is the dot product of the two vectors

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

and

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(3)+2(1)+3(5) & 1(4)+2(0)+3(2) \\ 3(3)+1(1)+2(5) & 3(4)+1(0)+2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 10 \\ 20 & 16 \end{bmatrix}.$$

Properties of Matrix Multiplication:

Let A, B, C be matrices, and k a scalar. Then, whenever the products and sums are defined, we have

1. Associativity of Matrix Multiplication
$$(AB)C = A(BC)$$
2. Left Distributive Property
$$A(B + C) = (AB) + (AC)$$
3. Right Distributive Property
$$(B + C)A = (BA) + (CA)$$
4. Commutativity of Scalar Multiplication
$$k(AB) = (kA)B = A(kB)$$

Note: We will observe the following order of operation:

$$(AB) + (CD) := AB + CD$$

for any matrices A, B, C, D where the above left hand side is defined.

Moreover, given two matrices A, B, AB may be defined but BA may not be. Also, even if AB and BA are defined, it is possible for

$$AB \neq BA,$$

that is, matrix multiplication is not a commutative operation.

The Transpose of a Matrix:

Given an $m \times n$ matrix

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

we define the **transpose** of A, denoted

$$A^T$$

to be the matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Example:

For

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & -1 \end{bmatrix}$$

we have

$$A^T = \begin{bmatrix} 1 & -4 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}.$$

Square Matrices:

An $n \times n$ matrix A is said to be a **square matrix.** That is a matrix of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is a square matrix.

On the set of $n \times n$ matrices with real entries, we have three operations: addition, scalar multiplication and matrix multiplication. The $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

is called the $n \times n$ **identity matrix**, because for any other $n \times n$ matrix A, we have

$$I_n A = A I_n = A.$$

The identity matrix In is an example of a **diagonal matrix.** A diagonal matrix is a matrix of the form

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

for any real numbers

$$d_1, d_2, \cdots, d_n.$$

Often

$$\text{diag}\{d_1, d_2, \cdots, d_n\}$$

is used to represent this diagonal matrix D.

Example:

The matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

is an example of a diagonal matrix.

A diagonal matrix of the form

$$\text{diag}\{k, k, \cdots, k\}$$

is said to be a **scalar matrix**. Note that

$$\text{diag}\{k, k, \cdots, k\} = kI_n$$

and thus

$$\text{diag}\{k, k, \cdots, k\}A = A\text{diag}\{k, k, \cdots, k\} = kA.$$

Given a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

the **diagonal** of the matrix A is the list of elements

$$a_{11}, a_{22}, \cdots, a_{nn}.$$

The **trace** of A, denoted tr(A), is the sum diagonal elements, that is

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{k=1}^{n} a_{kk}.$$

Some Properties of the Trace:

Given two $n \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ and k a scalar, we have

1. $$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$$

2. $$\text{tr}(A) = \text{tr}(A^T)$$

3. $$\text{tr}(kA) = k\text{tr}(A)$$

4. $$\text{tr}(AB) = \text{tr}(BA).$$

Example:

Let

$$A = \begin{bmatrix} 3 & 0 & 5 & 1 \\ 2 & -2 & 1 & \frac{1}{7} \\ \pi & 0 & 1 & \sqrt{2} \\ 0 & 0 & 3 & 2 \end{bmatrix}.$$

The diagonal of A is the elements

$$3, -2, 1, 2$$

and

$$\mathrm{tr}(A) = 3 + (-2) + 1 + 2 = 4.$$

Powers of Matrices and Polynomials in a Matrix Variable:

Given an $n \times n$ square matrix A, for any natural number m we may define

$$A^m = \underbrace{AA \cdots A}_{m \quad times}.$$

We also define

$$A^0 = I_n.$$

Moreover, we may take any polynomial with real coefficients

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_m x^m$$

and define

$$f(A) = a_o I_n + a_1 A + a_2 A^2 + \cdots a_m A^m.$$

This is said to be a **polynomial in a matrix variable**.

Example:

Given the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

we have:

$$A^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix},$$

and in general

$$A^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}.$$

For the polynomial

$$f(x) = 1 + 2x + 3x^3$$

we have

$$f(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 22 \\ 0 & 6 \end{bmatrix}.$$

Invertible or Nonsingular Matrices:

A square $n \times n$ matrix A is said to be **invertible** or **nonsingular** if there is a matrix B so that

$$AB = BA = I_n.$$

For example, consider

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Here we have

$$B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

satisfies

$$AB = BA = I_2.$$

Some Notes:

1. From the definition, a matrix B (if it exists) must also be an $n \times n$ matrix.

2. Suppose there are two matrices B, C which satisfy

   $$AB = BA = I_n, \quad AC = CA = I_n$$

   then it follows that $B = C$. To see this notice that

   $$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

   Thus, if A is invertible, then A^{-1} called the **inverse of** A, is the unique matrix which satisfies

   $$AA^{-1} = A^{-1}A = I_n.$$

We should note as well that not every matrix is invertible, for instance the matrix $0_{n,n}$ which we shall call 0_n for short, is the $n \times n$ matrix with all entries being zero. This matrix has the property that for any other square $n \times n$ matrix A, we have

$$0_n A = A 0_n = 0_n$$

and thus 0_n^{-1} cannot exist.

However, there are other square matrices who cannot have inverses. For instance

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

cannot have an inverse. To see this let

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

First we observe that

$$AB = A, \quad BA = 0_2.$$

If A^{-1} were to exist then

$$B = B(AA^{-1}) = 0_2 A^{-1} = 0_2$$

which is a contradiction.

Also, we note

$$AB \neq BA.$$

Invertible 2×2 Matrices:

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We define the **determinant** of A, denoted by

$$\det(A)$$

to be the quantity

$$\det(A) = ad - bc.$$

It turns out that A is invertible if and only if $\det(A) \neq 0$, and in such a case

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example: The matrix

$$A = \left[\begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array} \right]$$

has

$$\det(A) = 1(2) - 2(-1) = 4 \neq 0$$

and thus

$$A^{-1} = \frac{1}{4} \left[\begin{array}{cc} 2 & -2 \\ 1 & 1 \end{array} \right].$$

Special Types of Matrices:

We have considered general $n \times m$ matrices as well as $n \times n$ (square) matrices. In the class of square matrices, we discussed diagonal and scalar matrices – which are a certain type of diagonal matrices. There are other important types of square matrices which we can consider.

A square matrix A is said to be an **upper triangular matrix** if it is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

A square matrix A is said to be an **lower triangular matrix** if it is of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

A matrix A is said to be a **triangular matrix** if it is either an upper triangular matrix or a lower triangular matrix.

A matrix A is said to be **symmetric** if

$$A^T = A.$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

is symmetric.

A square matrix A is said to be **orthogonal** if

$$A^{-1} = A^T.$$

The following are examples of orthogonal matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \theta \in [0, 2\pi).$$