Math 311: Systems of Linear Equations

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Linear Equations:

A **linear equation** in the unknowns

$$x_1, x_2, x_3, x_4, \cdots, x_{n-1}, x_n$$

is an equation that can be put in the **standard form**

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_nx_n = b,$$

where $a_1, a_2, a_3, a_4, \cdots, a_{n-1}, a_n, b$ are constants, with the a_i's not all zero*.

A **solution** (or a **particular solution**) of the linear equation is a list of values

$$k_1, k_2, k_3, \cdots, k_{n-1}, k_n$$

satisfying

$$a_1k_1 + a_2k_2 + a_3k_3 + \cdots + a_{n-1}k_{n-1} + a_nk_n = b.$$

*If $a_1 = 0, a_2 = 0, \cdots, a_n = 0$ then the equation is **degenerate**, and in such a case if $b = 0$ then all values of $\vec{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ satisfy the given equation, and if $b \neq 0$ then no values of $\vec{x}$ will satisfy the equation.

We may express the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_nx_n = b$$

using matrix notation as

$$A\vec{x} = \vec{b},$$

where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix},$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b \end{bmatrix}.$$

Example:

Consider the linear equation

$$3x_1 + 2x_2 - x_3 = 4.$$

We have

$$x_1 = 1, \qquad x_2 = 1, \qquad x_3 = 1$$

is a solution to this equation. However

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = -4$$

is as well. Moreover, any values of the form

$$x_1 = x, \quad x_2 = y, \quad x_3 = 3x + 2y - 4,$$

for any real number x, y, is a solution.

When graphing the set of solutions in $\mathbb{R}^3$, one obtains a plane.

Example:

Consider the linear equation

$$x_1 - 2x_2 = 7.$$

We have that

$$x_1 = 7, \quad x_2 = 0$$

and

$$x_1 = 5, \quad x_2 = -1$$

are both solutions.

In general, for any real number x, we have

$$x_1 = x, \quad x_2 = \frac{1}{2}x - \frac{7}{2}$$

is a solution. The graph of all solutions constitute a line in $\mathbb{R}^2$.

In general, for the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

the graph of the set of solutions is a **hyper-plane** in $\mathbb{R}^n$. That is is the the $n-1$ dimensional flat object that is described uniquely by $n-1$ parameters. Assume for simplicity that $a_n \neq 0$, then the collection of points

$$(x_1, x_2, \cdots, x_{n-1}, -\frac{a_1}{a_n}x_1 - \frac{a_2}{a_n}x_2 - \cdots - \frac{a_{n-1}}{a_n}x_{n-1} + \frac{b}{a_n})$$

in $\mathbb{R}^n$, for any values of

$$x_1, x_2, x_3, \cdots, x_{n-1},$$

constitutes all solutions of the linear equation.

Systems of Linear Equations:

A **system of linear equations** is a list of linear equations with the same unknowns. In particular, a system of m linear equations in n unknowns $x_1, x_2, \cdots, x_n$ can always be put in the standard form (LS):

$$\begin{array}{cccccccc} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_2, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m, \end{array}$$

where a_{ij}, b_i are real numbers that satisfy the property that for each i, $a_{ij} \neq 0$ for some j.

The numbers a_{ij} are called the **coefficients** of the linear system (LS). The above system (LS) is called an $m \times n$ system, (read m by n system).

The system (LS) is said to be **homogeneous** if

$$b_1 = 0, \quad b_2 = 0, \quad \cdots \quad , b_m = 0.$$

A **solution** (or a **particular solution**) of the system (LS) is a list of values for the unknowns $x_1, x_2, \cdots, x_n$. The set of all solutions to the system (LS) is called the **solution set** or the **general solution** of the linear system.

If the linear system (LS) has no solutions, it is said to be **inconsistent**. Otherwise, it is called **consistent**.

Given another linear system (LS'):

$$
\begin{array}{cccccccc}
c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n = d_1, \\
c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + \cdots + c_{2n}x_n = d_2, \\
c_{31}x_1 + c_{32}x_2 + c_{33}x_3 + \cdots + c_{3n}x_n = d_2, \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
c_{r1}x_1 + c_{r2}x_2 + c_{r3}x_3 + \cdots + c_{rn}x_n = d_r,
\end{array}
$$

We say that (LS) and (LS') are **equivalent** if they have the same solution set.

Matrix representation of (LS):

We may expression the system (LS) in the form:

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}.$$

We note that in the case where A is a square matrix and invertible, the solution to (LS) will exist and is given by $\vec{x} = A^{-1}\vec{b}$.

We note that if $\vec{u}$ and $\vec{v}$ are two solutions to the non-homogeneous system

$$A\vec{x} = \vec{b},$$

then their difference

$$\vec{w} =: \vec{u} - \vec{v}$$

is a solution to the homogeneous system

$$A\vec{x} = \vec{0}.$$

To see this observe:

$$A\vec{w} = A(\vec{u} - \vec{v})$$

$$= A\vec{u} - A\vec{v}$$

$$= \vec{b} - \vec{b} = \vec{0}.$$

A second observation we now make is that for any two solutions $\vec{y}, \vec{z}$ to the homogeneous system

$$A\vec{x} = \vec{0},$$

any linear combination

$$c\vec{y} + k\vec{z}$$

is also a solution. To see this, we observe:

$$A(c\vec{y} + k\vec{z}) = cA\vec{y} + kA\vec{z}$$

$$= c\vec{0} + k\vec{0} = 0.$$

Thus, we may summarize these results in the following theorem:

Theorem: The set of solutions to the linear homogeneous system

$$A\vec{x} = 0$$

is closed under linear combinations. Moreover, given any solution $\vec{u}$ to the non-homogeneous system

$$A\vec{x} = \vec{b},$$

then any other solution $\vec{v}$ is of the form

$$\vec{v} = \vec{u} + \vec{x_h}$$

where $\vec{x_h}$ is a solution to the homogeneous solution $A\vec{x} = 0$. Also, If A is a square invertible matrix, $\vec{0}$ is the only solution to the homogeneous problem, and thus solutions to the non-homogeneous system are unique.

The Augmented Matrix for a Linear System:

Recall that the linear system (LS) may be represented by the matrix equation

$$A\vec{x} = \vec{b}$$

with $A, \vec{x}, \vec{b}$ as defined above; for this linear system (LS), we define the so-called **augmented matrix**, often denoted

$$[A \vdots \vec{b}],$$

which is given by

$$[A \vdots \vec{b}] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right].$$

Example:

Consider the linear system

$$3x_1 + x_2 - 2x_3 = 1$$

$$x_1 + x_2 + x_3 = 3.$$

A particular solution of the system is

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 2$$

and the solution set or general solution is given by

$$x_1 = \frac{3}{2}t - 1, \quad x_2 = 4 - \frac{5}{2}t, \quad x_3 = t$$

for any real number t.

Whereas, the linear system

$$3x_1 + x_2 - 2x_3 = 1$$

$$x_1 + x_2 + x_3 = 3$$

$$x_1 - x_2 - x_3 = 0$$

has its general solution being

$$x_1 = \frac{3}{2}, \qquad x_2 = -\frac{1}{6}, \qquad x_3 = \frac{5}{3}.$$

Equivalent Systems and Elementary Row Operations:

Recall that two systems (LS) and (LS') are said to be **equivalent** if they have the same solution set. At this point we want to examine operations that one can perform on a given linear system which preserve the solution set of the system.

Consider again the linear system (LS)

$$
\begin{array}{cccccccc}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1, \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2, \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_2, \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m.
\end{array}
$$

One way to obtain an equivalent system is to take non-zero constants

$$k_1, \quad k_2, \quad \cdots, k_m$$

and consider the system (LS")

$$k_1a_{11}x_1 + k_1a_{12}x_2 + \cdots + k_1a_{1n}x_n = k_1b_1,$$

$$k_2a_{21}x_1 + k_2a_{22}x_2 + \cdots + k_2a_{2n}x_n = k_2b_2,$$

$$k_3a_{31}x_1 + k_3a_{32}x_2 + \cdots + k_3a_{3n}x_n = k_3b_2,$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$k_ma_{m1}x_1 + k_ma_{m2}x_2 + \cdots + k_ma_{mn}x_n = k_mb_m.$$

If we drop the assumption that all the $k_n \neq 0$, then the systems (LS') may not be equivalent to (LS). However, any solution to (LS) will be a solution to (LS"). (WHY?)

Moreover, any solution to (LS) will satisfy the linear equation

$$(k_1a_{11}+\cdots+k_ma_{m1})x_1+\cdots+(k_1a_{1n}+\cdots+k_ma_{mn})x_n$$

$$= k_1b_1 + \cdots + k_mb_m,$$

which is obtained by taking the sum of the left hand sides of all equations in (LS") and setting that equal to the sum of the right hand sides of each equation in (LS"). Such an equation is called a **linear combination** of the equations in the system (LS).

Theorem: Two linear systems (LS) and (LS') are equivalent if and only if any equation in (LS') may be obtained by a linear combination of the equations in (LS), and vice versa.

To see why this theorem is true, we will first discuss what are called **elementary row operations** for a given matrix A, which are as follows:

1. **Interchange Property:** Interchange any two rows of the matrix.

2. **Scaling Property:** Multiply every entry of some row of the matrix by the same nonzero scalar.

3. **Row Addition Property:** Add a multiple of one row of the matrix to another row of the matrix.

Notations we shall use to represent the above three elementary row operations:

1. Interchanging rows i and j of a matrix

$$R_i \leftrightarrow R_j$$

2. Multiplying row i by a scalar k

$$R_i \to kR_i$$

3. Replacing row i by row i plus k times row j

$$R_i \to R_i + kR_j$$

Matrix Representations of Elementary Row Operations:

Given an $m \times n$ matrix A, and I_m the $m \times m$ identity matrix. Let $I_{m,R_i \leftrightarrow R_j}$ be the matrix obtained from I_m by interchanging the ith and jth rows of I_m. Then

$$I_{m,R_i \leftrightarrow R_j} A$$

is the matrix obtained by $R_i \leftrightarrow R_j$ applied to A. Moreover $I^2_{m,R_i \leftrightarrow R_j} = I_m$.

Let $I_{m,R_i \to kR_i}$ be the matrix obtained from I_m by replacing the ith row of I_m by k times the ith row of I_m. Then

$$I_{m,R_i \to kR_i} A$$

is the matrix obtained by $R_i \to kR_i$ applied to A. Moreover $I_{m,R_i \to kR_i} I_{m,R_i \to \frac{1}{k} R_i} = I_m$.

Let $I_{m,R_i \to R_i + kR_j}$ be the matrix obtained from I_m by replacing the ith row of I_m by the ith row plus k times the jth row of I_m. Then

$$I_{m,R_i \to R_i + kR_j} A$$

is the matrix obtained by $R_i \to R_i + kR_j$ applied to A. Moreover $I_{m,R_i \to R_i + kR_j} I_{m,R_i \to R_i - kR_j} = I_m$.

We will use these operations of the augmented matrix $[A \vdots \vec{b}]$ for a given system $A\vec{x} = \vec{b}$. We observe that these elementary row operations applied to the augmented matrix results in a new matrix which is an augmented matrix for an equivalent system. This we shall now prove:

Suppose that

$$A\vec{x} = \vec{b}, \qquad C\vec{x} = \vec{d}$$

are two systems where $[A \vdots \vec{b}]$ may be transformed into $[C \vdots \vec{d}]$ by elementary row operations. Let $\vec{u}$ be a solution to $A\vec{x} = \vec{b}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ and $A_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$, is the ith row of A.

Because $A\vec{u} = \vec{b}$ we have

$$A_1 \cdot \vec{u} = b_1, A_2 \cdot \vec{u} = b_2, \cdots, A_m \cdot \vec{u} = b_m.$$

Clearly if we interchange the equations $A_i \cdot \vec{u} = b_i$ and $A_j \cdot \vec{u} = b_j$ then the resulting system has the same solutions, since it is the same set of equations, simply stated in a different order. Moreover, this corresponds with interchanging rows i and j in the augmented matrix $[A \vdots \vec{b}]$. Thus, the interchange property clearly preserves solutions.

Next, suppose that we multiply equation k $A_k \cdot \vec{u} = b_k$ by a scalar $r \neq 0$, then the resulting new system clearly has the same solutions since $(rA_k) \cdot \vec{u} = rb_k$ and $A_k \cdot \vec{u} = b_k$ are equivalent equations. The resulting system has an augmented matrix whose kth row is r times the kth row of the original augmented matrix. Thus we see that the scaling operation preserves solutions.

For the third type of row operation, the row addition property, we observe that if $\vec{u}$ satisfies

$$A_1 \cdot \vec{u} = b_1, \cdots, A_m \cdot \vec{u} = b_m$$

–which has the corresponding augmented matrix $[A \vdots \vec{b}]$ – then by considering the kth and pth equations $A_k \cdot \vec{u} = b_k$ and $A_p \cdot \vec{u} = b_p$, we see that $\vec{u}$ satisfies

$$(A_k + rA_p) \cdot \vec{u} = A_k \cdot \vec{u} + rA_p \cdot \vec{u} = b_k + rb_p.$$

If we replace the kth row by this new equation the corresponding system has the same solutions, and the new augmented matrix has its kth row being the kth row of $[A \vdots \vec{b}]$ plus r times the pth row of $[A \vdots \vec{b}]$.
Q.E.D.

Gaussian Elimination:

To solve a given system (LS), we will use the elementary row operations to produce a sequence of equivalent linear systems, with the final linear system having its augmented matrix in either **row echelon form** or **row-reduced echelon form.**

A **leading entry** for a row in a matrix is the first non-zero entry in the row, when going from left to right.

A matrix A is said to be in **row echelon form** (ref) if it satisfies the following three conditions:

1. Each non-zero row lies above every zero row.

2. The leading entry of a non-zero row lies in a column to the right of the column containing the leading entry of any preceding row.

3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.

If a matrix is **row-reduced echelon form** (rref) it also satisfies:

4. If a column contains the leading entry of some row, then all other entries of that column are zero.

5. The leading entry of each non-zero row is 1.

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Given a linear system (LS) with augmented matrix $[A \vdots \vec{b}]$, in order to solve the system (LS), our strategy is to transform the augmented matrix into row echelon – or better yet, row-reduced echelon form – by a sequence of elementary row operations. This algorithm goes by the name of **Gaussian elimination**.

A matrix A is said to be **row equivalent** to a matrix B, written

$$A \sim B$$

if B can be obtained from A by a sequence of elementary row operations.

By the nature of these operations, we have that $\sim$ is an equivalence relation of the set of $m \times n$ matrices, that is:

1. Reflexivity

$$A \sim A$$

2. Symmetry

$$A \sim B \Rightarrow B \sim A$$

3. Transitivity

$$A \sim B \quad and \quad B \sim C \Rightarrow A \sim C$$

Thus we see that the linear systems with similar augmented matrices will have the same solution set, which gives a justification why Gaussian elimination is used to solve linear systems.

Example: Find the row-reduced echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ -3 & -6 & -6 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix}.$$

Solution:

Noticing that row 2 is -3 times row 1 we perform the following:

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ -3 & -6 & -6 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \longrightarrow_{R_2 \to R_2 + 3R_1}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$$

Then, we interchange rows 2 and 3 to get:

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \longrightarrow_{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, we want column 1 to have a 1 in the top entry and 2 0's below the one, thus:

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow_{R_2 \to R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, we want the leading entry in row 2 to be 1. Thus, we perform the following:

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow_{R_2 \to -R_2}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This final matrix is in row echelon form.

Next, we want the second column to contain all zeros except the entry of 1, which is the leading entry in row 2.

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow_{R_1 \to R_1 - 2R_2}$$

$$\begin{bmatrix} 1 & 0 & -6 & 5 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-reduced echelon form.

Example: Find the general solution to the linear system

$$x + 2y + 2z = -1,$$

$$-3x - 6y - 6z = 3,$$

$$2x + 3y = 1.$$

Solution: The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & -1 \\ -3 & -6 & -6 & 3 \\ 2 & 3 & 0 & 1 \end{array}\right]$$

and the row-reduced echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & 5 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix is an augmented matrix for the system

$$x - 6z = 5,$$

$$y + 4z = -3.$$

The solution to this system is

$$x = 6t + 5, \quad y = -4t - 3, \quad z = t$$

for any real number t.

The **rank** of a matrix A, denoted by

$$\text{rank}(A),$$

is the number of nonzero rows in the row-reduced echelon form of A.

An observation we shall make about the solutions to a linear system

$$A\vec{x} = \vec{b}$$

for A an $m \times n$ matrix is:

1. If

$$\text{rank}(A) \neq \text{rank}([A \vdots \vec{b}])$$

then the system is inconsistent, and thus there is no solution to the linear system $A\vec{x} = \vec{b}$.

2. If

$$\text{rank}(A) = \text{rank}([A \vdots \vec{b}]) = n$$

then the system $A\vec{x} = \vec{b}$ has a unique solution.

3. If

$$\text{rank}(A) = \text{rank}([A \vdots \vec{b}]) < n,$$

then the system $A\vec{x} = \vec{b}$ has a infinitely many solutions. Moreover, in the general form of a solution, there are $n - \text{rank}(A)$ free variables/parameters.

Example: Find the solution to the linear system, if it exists.

$$x - y + 2z + 3w = 2,$$

$$2x + y + z = 1,$$

$$x + 2y - z - 3w = 7.$$

Solution: The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -3 & 7 \end{array}\right].$$

We will now find the row-reduced echelon form of this matrix.

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -3 & 7 \end{array}\right] \longrightarrow_{R_2 \to R_2 - 2R_1}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 3 & -3 & -6 & -3 \\ 1 & 2 & -1 & -3 & 7 \end{array}\right] \longrightarrow_{R_3 \to R_3 - R_1}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 3 & -3 & -6 & -3 \\ 0 & 3 & -3 & -6 & 5 \end{array}\right] \longrightarrow_{R_3 \to R_3 - R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 3 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right] \longrightarrow_{R_3 \to \frac{1}{8} R_3}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 3 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow_{R_2 \to \frac{1}{3} R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow_{R_1 \to R_1 + R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow_{R_1 \to R_1 - R_3, R_2 \to R_2 + R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The above matrix is in row-reduced echelon form, and thus the augmented matrix for our original system has rank 3. However, the row-reduced echelon form for

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & -3 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and thus rank$(A) = 2$. Hence our system is inconsistent.

Example: Find the general form of a solution to the linear system, if it exists:

$$x + 2y + 3z = 9,$$

$$2x - y + z = 8,$$

$$3x - z = 3.$$

Solution: The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right].$$

We now proceed to find the row-reduced echelon form of this matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \longrightarrow_{R_3 \to R_3 - 3R_1, R_2 \to R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right] \longrightarrow_{R_2 \to \frac{-1}{5} R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right] \xrightarrow{}_{R_3 \to R_3 + 6R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \xrightarrow{}_{R_3 \to \frac{-1}{4} R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{}_{R_2 \to R_2 - R_3, R_1 \to R_1 - 3R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{}_{R_1 \to R_1 - 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Thus, the unique solution to the linear system is

$$x = 2, \quad y = -1, \quad z = 3.$$

Calculating the Inverse of a Square Matrix:

Given an $n \times n$ matrix A, we would like to know an algorithm for computing the inverse of A whenever it exists. An observation we shall make is that A *is invertible if and only if the row-reduced echelon form of* A *is* I_n. To see this, suppose that applied to A and I_n is the row-reduced echelon form of A. Let

$$E_1, E_2, E_3, \cdots, E_r$$

be the corresponding matrices that are applied by multiplication on the left, to result in performing these elementary row operations. Then we have

$$E_r \cdots E_3 E_2 E_1 A = I_n.$$

Then the product of the matrices

$$E_r \cdots E_3 E_2 E_1$$

is A^{-1}.

Thus, our procedure to find A^{-1} for a given matrix A is as follows:

1. Consider the $n \times 2n$ matrix formed from A and I_n where the first A columns going from left to right are A and the last n columns are I_n. We denote this matrix by

$$[A \vdots I_n].$$

2. Find the row-reduced echelon form of $[A \vdots I_n]$.

3. If the first n columns, when going from left to right, are I_n, the the last n columns are A^{-1}. That is, if the row-reduced echelon form of $[A \vdots I_n]$ is $[I_n \vdots B]$, then $B = A^{-1}$.

4. If the first n columns of the row-reduced echelon form of $[A \vdots I_n]$ is not I_n, then A^{-1} does not exist.

Example: Given

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

find A^{-1} if it exists.

Solution:

$$[A \vdots I_2] = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow_{R_1 \leftrightarrow R_2}$$

$$[A \vdots I_2] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

which is in row-reduced echelon form. Noting that the first two columns are not I_2, we see that A is not invertible.

Example: Given

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & -5 & 7 \end{bmatrix},$$

find A^{-1} if it exists.

$$[A \vdots I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 3 & -5 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{} {}_{R_2 \to R_2 - R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 3 & -5 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{} {}_{R_2 \to \frac{-1}{2} R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 3 & -5 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{} {}_{R_1 \to R_1 - R_2, R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & -5 & 7 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \xrightarrow{} R_2 \to R_2 - 3R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -5 & 7 & -\frac{3}{2} & -\frac{3}{2} & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \xrightarrow{} R_2 \to R_2 - 7R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -5 & 0 & -5 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \xrightarrow{} R_2 \to -\frac{1}{5}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -\frac{2}{5} & -\frac{1}{5} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}.$$