Math 311: Determinants, Eigenvalues and Eigenvectors

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The Determinant of a Square Matrix:

We would like to define a function

$$\det : M_{n,n}(\mathbb{F}) \to \mathbb{F},$$

that satisfies the following properties:

1. det is a linear function as a function of each of the rows of the matrix.

   That is, if we think of the matrix A as a collection of rows $(R_1, R_2, \cdots, R_n)$, then:

   $$\det(R_1, \cdots, cR_i + R_i', \cdots, R_n)$$

   $$= c\det(R_1, \cdots, R_i, \cdots, R_n)$$

   $$+\det(R_1, \cdots, R_i', \cdots, R_n).$$

2. If any of the rows of A are equal, then

$$\det(A) = 0.$$

3.

$$\det(I_n) = 1$$

It turns out that such a function does exist, and is unique.

Properties of the Determinant Function*:

1.

$$\det(AB) = \det(A)\det(B)$$

and thus,

$$1 = \det(I_n) = \det(A)\det(A^{-1}),$$

which implies

$$\det(A^{-1}) = \det(A)^{-1}.$$

2.

$$\det(A) \neq 0 \iff A \quad \text{invertible.}$$

*that follow for free

A Formula for the Determinant:

A bijection from the set of integers

$$\{1, 2, \cdots, n\}$$

to itself is called a **permutation** of the set $\{1, 2, \cdots, n\}$. The collection of all permutations, denoted S_n, is an algebraic group under the operation of composition. This collection contains $n!$ elements.

Permutations can be classified as **odd** or **even**, according to the parity of the number of **transpositions** need to decompose the permutation into a product of transpositions. Here a transposition is a permutation that fixes all but two of the numbers, and exchanges two numbers.

If $\sigma \in S_n$, then the **sign** of σ, denoted sgnσ, is 1 if σ is even, and -1 if σ is odd.

Thus, for an $n \times n$ matrix $A = (a_{ij})$, we have

$$\text{det}(A) = \sum_{\sigma \in S_n} \text{sgn}\sigma \cdot a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

A Specific Formula for the Determinant of a 2×2 Matrix:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

is the determinant of A.

A Specific Formula for the Determinant of a 3×3 Matrix:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Examples:

Find the determinants of the following matrices.

1.
$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$
$$\det(A) = 0 \cdot 1 - 0 \cdot 1 = 0$$

2.
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\det(A) = 0 \cdot 1 - 1 \cdot 1 = -1$$

3.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2$$

4.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det(A) = \frac{1}{2} \cdot \frac{1}{2} - \frac{-\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = 1$$

5.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det(A) = \frac{1}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{-\sqrt{3}}{2} = 1$$

6.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\det(A)$$

$$= \frac{1}{2} \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{vmatrix} - \frac{-\sqrt{3}}{2} \begin{vmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & -1 \end{vmatrix} + 0 \begin{vmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 \end{vmatrix}$$

$$= \frac{1}{2} \cdot \frac{-1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{-\sqrt{3}}{2} + 0 = -1$$

7.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 7 \\ -2 & 2 & -2 \end{bmatrix}$$

$$\det(A)$$

$$= 1 \cdot \begin{vmatrix} 5 & 7 \\ 2 & -2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 7 \\ -2 & -2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 5 \\ -2 & 2 \end{vmatrix}$$

$$= 1(-10-14)-2(-6+14)-1(6+10) = -56.$$

Eigenvalues and Eigenvectors:

Given a linear transformation

$$L : \mathbb{R}^n \to \mathbb{R}^n,$$

we say that $\lambda \in \mathbb{R}$ (or more generally $\mathbb{C}$) is an **eigenvalue** for L if there exists a non-zero vector $\vec{x} \in \mathbb{R}^n$ so that

$$L(\vec{x}) = \lambda \vec{x}.$$

Note that such an $\vec{x}$ is called an **eigenvector** for L corresponding to the eigenvalue λ.

As we saw earlier, linear transformations $L : \mathbb{R}^n \to \mathbb{R}^n$ can be expressed uniquely with respect to the usual basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbb{R}^n$ via

$$L(\vec{x}) = A\vec{x}.$$

Thus, given a matrix $A \in M_{n,n}(\mathbb{R})$ we say that $\lambda \in \mathbb{R}$ (or more generally $\mathbb{C}$) is an **eigenvalue** for A if there exists a non-zero $\vec{x} \in \mathbb{R}^n$ so that

$$A\vec{x} = \lambda\vec{x}.$$

Once again, we call such an $\vec{x}$ an **eigenvector** for A corresponding to the eigenvalue λ.

Note that by properties of L or by properties of scalar multiplication, if $\vec{x}$ is an eigenvector for L or for A, corresponding to an eigenvalue λ, then so is $c\vec{x}$ for any $c \neq 0$ in $\mathbb{R}$.

Hence, we define the **eigenspace** for the eigenvalue λ to be the subspace generated by the span of eigenvectors corresponding to an eigenvalue λ. Such a subspace is invariant under the linear transformation L or under the multiplication by A.

Observe that

$$A\vec{x} = \lambda\vec{x},$$

is equivalent to

$$(A - \lambda I_n)\vec{x} = \vec{0}.$$

Thus we see that

$$\det(A - \lambda I_n) = 0$$

if and only if λ is an eigenvalue of A.*

We note that the polynomial $\det(A - \lambda I_n)$ is called the **characteristic polynomial** for A. Moreover, the eigenvalues for A are the so-called roots of the **characteristic equation**

$$\det(A - \lambda I_n) = 0.$$

We note as well that the collection of eigenvalues of A is called the **spectrum** of A.

*Because we are requiring that the operator $A - \lambda I$ has a non-trivial kernel whenever λ is an eigenvalue, and hence is not invertible whenever λ is an eigenvalue.

Example: Find all eigenvalues for A and corresponding eigenvectors for each eigenvalue, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}.$$

Solution: We consider the characteristic equation

$$\det(A - \lambda I_2)$$

$$= \det\left(\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \begin{vmatrix} 1-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-2-\lambda) = 0.$$

Thus, the eigenvalues are $1, -2$.

We now substitute the λ values into $(A - \lambda I_n)\vec{x} = \vec{0}$ to find corresponding eigenvectors.

Case $\lambda = 1$:

We need to consider

$$\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This implies $y = 0$ and thus

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is an eigenvector for the eigenvalue 1.

Case $\lambda = -2$:

We need to consider

$$\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This implies $y = -3x$ and thus

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

is an eigenvector for the eigenvalue -2.