Sequences and Series

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Sequences and Limits of Sequences:

A **sequence** of real numbers is a function f from the natural numbers into the real numbers, that is

$$f : \mathbb{N} \to \mathbb{R}.$$

Usually, we write f_n to represent $f(n)$, and it is common to write the function values

$$f_1, f_2, f_3, \cdots, f_n, \cdots$$

to represent the sequence, and we often abbreviate this by

$$(f_n) \quad or \quad (f_n : n \in \mathbb{N}).^*$$

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*We prefer $(f_n : n \in \mathbb{N})$ instead of $\{f_n : n \in \mathbb{N}\}$, because the later is the range of the sequence.

There are several ways which a sequence may be defined. Sometimes a sequence is defined by a formula for f_n in terms of n. Sequences may also be defined **inductively** or **recursively**. That is we calulate f_n in terms of $f_1, f_2, \cdots, f_{n-1}$.

Some Examples of Sequences:

The sequence $f_n := 1$ is a constant sequence, since each value in the sequence is 1.

The sequence $f_n := \frac{1}{n}$ is the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \cdots.$$

The sequence defined by $f_1 = 1$, $f_2 = 1$, and for $n \geq 3$

$$f_n = f_{n-1} + f_{n-2}$$

is the so-called Fibonacci sequence; it's terms are given by

$$1, 1, 2, 3, 5, 8, 13, 21, \cdots.$$

The sequence defined by $f_n = (-1)^n$, has terms given by

$$-1, 1, -1, 1, -1, 1, \cdots.$$

Limits of Sequences or Real Numbers:

A sequence or real numbers (f_n) is said to **converge** to $L \in \mathbb{R}$, and L is said to be the **limit** of the sequence, if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ so that for $n \geq N$ we have $|f_n - L| < \varepsilon$. Moreover, we often denote L by

$$\lim_{n \to \infty} f_n.$$

If a sequence (f_n) has a limit $\lim_{n \to \infty} f_n$, then we say that (f_n) is a **convergent** sequence. If no limit exists, then (f_n) is said to be **divergent.**

In our first example of a sequence $f_n = 1\ \forall n \in \mathbb{N}$, we have that $\lim_{n\to\infty} f_n = 1$ since given any $\varepsilon > 0$

$$0 = |1 - 1| = |f_n - 1| < \varepsilon \quad \forall n \in \mathbb{N}.$$

In our example $f_n = \frac{1}{n}$, we have that $\lim_{n\to\infty} f_n = 0$. To see this, let $\varepsilon > 0$ be given. By the Archimedean Property we may pick $N \in \mathbb{N}$ so that

$$N > \frac{1}{\varepsilon}.$$

Thus, in such a case, for $n \geq N$ we have

$$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

The Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \cdots$$

is divergent because $f_n \geq n - 1$ for all n. We shall now prove this by the principle of complete induction:

If $n = 1$, clearly $1 = f_1 \geq 0$. If $n = 2$ then $f_2 = 1 \geq 2-1 = 1$. If $n = 3$ then $f_3 = 2 \geq 2-1$. Now suppose for some $n \in \{3, 4, 5, 6, \cdots\}$ we have $\forall k \in \{1, 2, 3, \cdots, n\}$ that $f_k \geq k - 1$. We must show that $f_{n+1} \geq n$.

Now,

$$f_{n+1} = f_n + f_{n-1} \geq n-1+n-2 \geq n-1+3-2 = n.$$

Thus we see that $f_n \geq n - 1$. Hence, we see that if $\lim_{n\to\infty} f_n$ exists and equal L, then by the Archimedean property, there is an $N \in \mathbb{N}$ so that $N \geq L + 2$ Thus for $n \geq N$ we have

$$f_n \geq n - 1 \geq N - 1 \geq L + 1.$$

Hence,

$$|f_n - L| = f_n - L \geq 1$$

and thus for $\varepsilon < 1$ we cannot obtain $|f_n - L| < \varepsilon$ when n is large enough.

Finally, for the sequence $f_n = (-1)^n$, we claim that $\lim_{n\to\infty} f_n$ does not exist. To see this we simply fix any $\varepsilon < \frac{1}{3}$. Then for any $n \in \mathbb{N}$, either $|(-1)^n - L| > \frac{1}{3}$ or $|(-1)^{n+1} - L| > \frac{1}{3}$.

Uniqueness of Limits of Sequences:

If $\lim_{n\to\infty} f_n$ exists, it is a unique value.

Proof: Suppose L, M are limits of (f_n), then given $\varepsilon > 0$ there exists an $N, K \in \mathbb{N}$ so that $n \geq N$ we have

$$|f_n - L| < \frac{\varepsilon}{2}.$$

If $n \geq K$ then we have

$$|f_n - M| \leq \frac{\varepsilon}{2}.$$

Thus for $n \geq \max\{N, K\}$ we have

$$|L - M| = |L - f_n + f_n - M|$$

$$\leq |L - f_n| + |f_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

Consider the sequence

$$a_n = \frac{2n+1}{3n-2}, \qquad n \in \mathbb{N}.$$

We will now show

$$\lim_{n\to\infty} a_n = \frac{2}{3}.$$

To do this, we assume that we are given $\varepsilon > 0$ and we need to produce an $N \in \mathbb{N}$ so that $n \geq N$ implies

$$|\frac{2n+1}{3n-2} - \frac{2}{3}| < \varepsilon.$$

Now,

$$|\frac{2n+1}{3n-2} - \frac{2}{3}| = |\frac{2n+1}{3n-2} - \frac{2}{3} \cdot \frac{n-\frac{2}{3}}{n-\frac{2}{3}}|$$

$$= |\frac{2n+1}{3n-2} - \frac{2n-\frac{4}{3}}{3n-2}|$$

$$= |\frac{2n+1-(2n-\frac{4}{3})}{3n-2}|$$

$$= |\frac{\frac{7}{3}}{3n-2}|$$

$$= \frac{7}{3} \cdot \frac{1}{3n-2}$$

$$< \frac{7}{3} \cdot \frac{1}{3(n-1)} = \frac{7}{9} \cdot \frac{1}{n-1}.$$

We know $\frac{1}{n} \to 0$ as $n \to \infty$ and thus $\frac{1}{n-1} \to 0$ as $n \to \infty$*. Thus we can make $\frac{1}{n} < \frac{9}{7}\varepsilon$ for $n > N$, for some $N \in \mathbb{N}$.

*→ is read "approaches"

A sequence

$$(a_n : n \in \mathbb{N})$$

converges if and only if

$$(a_{n+k} : n \in \mathbb{N})$$

converges for any fixed $k \in \mathbb{N} \cup \{0\}$. That is, for a sequence to converge, we only need to examine the so-called **tail end** of the sequence.

Theorem: Let (x_n) be a sequence and $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with limit 0, $C > 0$ is a constant, and for some $m \in \mathbb{N}$ we have

$$|x_n - x| \leq C a_n$$

then

$$\lim_{n \to \infty} x_n = x.$$

Proof: Let $\varepsilon > 0$ be given. Then there exists $K \in \mathbb{N}$ so that for $n \geq K$ we have

$$a_n = |a_n - 0| < \frac{\varepsilon}{C}$$

then for $n \geq \max\{K, m\}$ we have

$$|x_n - x| \leq C a_n < C \cdot \frac{\varepsilon}{C} = \varepsilon.$$

Q.E.D.

Example: For example, for $c \in (0,1)$ we will show

$$\lim_{n\to\infty} c^n = 0.$$

For $c \in (0,1)$ we write

$$c = \frac{1}{a+1}, \quad where \quad a := \frac{1}{c} - 1.$$

Then

$$0 < c^n = \frac{1}{(a+1)^n} \leq \frac{1}{1+na} < \frac{1}{na}.$$

Now

$$\frac{1}{n} \to 0$$

and is a positive sequence, so we use the above theorem to get $\lim_{n\to\infty} c^n = 0$.

Example: For $c > 0$ we claim that

$$(c^{\frac{1}{n}} : n \in \mathbb{N})$$

has limit 1.

To see this we examine cases. If $c = 1$ then $1^{\frac{1}{n}} \equiv 0$ for all n. So of course, the limit is 1.

If $c > 1$, then we know $c^{\frac{1}{n}} = 1 + d_n$ with $d_n > 0$. (Why?)

Thus,

$$c = (1 + d_n)^n \geq 1 + nd_n.$$

Hence $c - 1 \geq nd_n$, and so $d_n \leq \frac{c-1}{n}$. Consequently,

$$|c^{\frac{1}{n}} - 1| = d_n \leq (c - 1)\frac{1}{n},$$

and thus the result follows.

In the case $0 < c < 1$, we write $c^{\frac{1}{n}} = \frac{1}{1+h_n}$ with $h_n > 0$. Thus

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+hn_n} < \frac{1}{nh_n}.$$

It follows that

$$0 < h_n < \frac{1}{nc}$$

and thus we have

$$|c^{\frac{1}{n}} - 1| < \frac{1}{c} \cdot \frac{1}{n}.$$

Thus the result follows.

Example: We will show that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

To see this, we note that $n^{\frac{1}{n}} > 1$. Thus, $n^{\frac{1}{n}} = 1 + k_n$ for some $k_n > 0$. Hence

$$n = (1 + k_n)^n$$

$$= 1+nk_n+\frac{n(n-1)}{2}k_n^2+\cdots+k_n^n \geq 1+\frac{n(n-1)}{2}k_n^2.$$

It follows that $n - 1 \geq \frac{n(n-1)}{2}k_n^2$, and thus $k_n^2 \leq \frac{2}{n}$.

If $\varepsilon > 0$, it follows from the Archimedean Property that there is an $N \in \mathbb{N}$ so that $\frac{2}{N} < \varepsilon^2$. Hence, for $n \geq \max\{2, N\}$ we have $\frac{2}{n} < \varepsilon^2$. Therefore

$$0 < n^{\frac{1}{n}} - 1 = k_n \leq (\frac{2}{n})^{\frac{1}{2}} < \varepsilon.$$

Homework Exercises:

pages 59-60 numbers 1 a,c; 2 a,c; 3 a,b,c; 4; 5 a,c; 12; 14.

Limit Theorems:

A sequence of real numbers (x_n) is said to be **bounded**, if there exists an $M \in \mathbb{R}$ so that

$$|x_n| \leq M \qquad \forall n \in \mathbb{N}.$$

Convergent Sequences are Bounded: To see this suppose (x_n) is a convergent sequence with $\lim_{n\to\infty} x_n = x$. Then for $n \geq N$ we have

$$|x_n - x| < 1,$$

since (x_n) converges. Thus, for $n \geq N$ we have

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| \leq 1 + |x|.$$

Thus, if we take

$$M = \sup\{|x_1|, |x_2|, \cdots, |x_N|, 1 + |x|\}$$

we have

$$|x_n| \leq M.$$

The contrapositive of this statement – which is logically equivalent – is:

Unbounded Sequences Diverge.

Given two sequences (x_n) and (y_n) of real numbers, if

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y,$$

then the following also hold:

- $$x_n \pm y_n \to x \pm y \quad as \quad n \to \infty$$

- $$x_n y_n \to xy \quad as \quad n \to \infty.$$

- For $x \neq 0$ and n sufficiently large so $\frac{1}{x_n}$ is defined we have
$$\frac{1}{x_n} \to \frac{1}{x} \quad as \quad n \to \infty.$$

- For $y \neq 0$ and n sufficiently large so that $\frac{x_n}{y_n}$ is defined we have

$$\frac{x_n}{y_n} \to \frac{x}{y}.$$

Proof:

- Use

$$|x_n \pm y_n - (x \pm y)| \leq |x_n - x| + |y_n - y|.$$

  Each of the terms $|x_n - x|, |y_n - y|$ may be made arbitrarily small since $x_n \to x$ and $y_n \to y$.

- Use

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy|$$

$$\leq |x_n||y_n - y| + |y||x_n - x|,$$

x_n is bounded and $|x_n - x|, |y_n - y|$ can be made arbitrarily small.

- Use
$$|\frac{1}{x_n} - \frac{1}{x}| = \frac{|x_n - x|}{|x_n||x|}.$$
Then use x_n is bounded and $\frac{1}{|x_n|} \leq \frac{2}{|x|}$ to get the rest.

- For the last one use the second and third properties together.

The next result we would like to note is:

If $x_n \geq 0$ and $\lim_{n\to\infty} x_n$, then $\lim_{n\to\infty} x_n \geq 0$.

To see this, we suppose the contrary. If $\lim_{n\to\infty} x_n < 0$, then using $\varepsilon = \frac{|\lim_{n\to\infty} x_n|}{2}$ we have that for $n \geq N$

$$|x_n - \lim_{n\to\infty} x_n| < \frac{|\lim_{n\to\infty} x_n|}{2}$$

and thus

$$x_n \leq \frac{\lim_{n\to\infty} x_n}{2} < 0,$$

which is a contradiction.

Some corollaries of this are:

Corollary: $x_n \to x$, $y_n \to y$ and $x_n \leq y_n$ implies $x \leq y$.

To see this define the sequence $y_n - x_n$ and use the above result.

Corollary: If (x_n) is a bounded sequence which converges to x and $a \leq x_n \leq b$, then

$$a \leq x \leq b.$$

To see this use $x_n - a$ and $b - x_n$ and apply the above result.

The Squeeze Theorem: If $(x_n), (y_n), (z_n)$ are sequences satisfying

$$x_n \leq y_n \leq z_n$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = w$$

then

$$\lim_{n\to\infty} y_n = w.$$

To see this we observe that $|x_n - w|$ and $|z_n - w|$ go to 0 as $n \to \infty$. Moreover,

$$-|x_n - w| \leq x_n - w \leq y_n - w \leq z_n - w \leq |z_n - w|.$$

Another useful result is the following:

$$\lim_{n\to\infty} x_n = x \quad \Rightarrow \quad \lim_{n\to\infty} |x_n| = |x|.$$

To see this we note that

$$||x_n| - |x|| \leq |x_n - x|$$

and the term on the right side of the inequality $\to 0$ as $n \to \infty$.

Finally $x_n \geq 0$ and $x_n \to x$ implies $\sqrt{x_n} \to \sqrt{x}$.

To see this note that

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}}|x_n - x|.$$

Homework: Pages 67-68 exercises 1 b,c,d; 2; 3; 6 (turn in); 8 (turn in); 13 (turn in); 18 a,b (turn in); 21 (turn in).

Problems are due on Thursday November 20th.

Monotone Sequences:

A sequence (x_n) is said to be **(strictly) increasing** if for each $n \in \mathbb{N}$ we have

$$(x_n < x_{n+1}) \quad x_n \leq x_{n+1}.$$

A sequence (x_n) is said to be **(strictly) decreasing** if for each $n \in \mathbb{N}$ we have

$$(x_n > x_{n+1}) \quad x_n \geq x_{n+1}.$$

A sequence (x_n) is said to be **monotone** if it is either increasing or decreasing.

Monotone Convergence Theorem:

If (x_n) is a bounded increasing sequence, then

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

If (x_n) is a bounded decreasing sequence, then

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

Proof: Let (x_n) be a bounded monotone sequence. By otherwise using $(-x_n)$ and page 38 exercise 5, we may assume that (x_n) is increasing; and in such a case we know $x := \sup\{x_n : n \in \mathbb{N}\}$ exists.

Let $\varepsilon > 0$ be given. Then $x - \varepsilon$ is not an upper bound for

$$S := \{x_n : n \in \mathbb{N}\}.$$

Hence, there is some $N \in \mathbb{N}$ so that

$$x - \varepsilon < x_N \leq x.$$

Since (x_n) is increasing and x is an upper bound for S we know that for all $m \geq N$ in $\mathbb{N}$, we have

$$x - \varepsilon < x_m \leq x.$$

Thus

$$|x_m - x| < \varepsilon$$

for $m \geq N$.
Q.E.D.

Example: Consider the sequence

$$x_n = \frac{1}{\sqrt[3]{n+1}}.$$

We will show that (x_n) is decreasing and bounded. Moreover, $x_n \geq 0 \ \forall n \in \mathbb{N}$ and we will show $\inf\{x_n : n \in \mathbb{N}\} = 0$.

We know that $\sqrt[3]{x}$ is increasing as a function, that is

$$a \leq b \quad \Longleftrightarrow \quad \sqrt[3]{a} \leq \sqrt[3]{b}.$$

Moreover,

$$\begin{aligned} 2 \geq 1 &\Longleftrightarrow n+2 \geq n+1 \\ &\Longleftrightarrow \sqrt[3]{n+2} \geq \sqrt[3]{n+1} \\ &\Longleftrightarrow \frac{1}{\sqrt[3]{n+1}} \geq \frac{1}{\sqrt[3]{n+2}} \\ &\Longleftrightarrow x_n \geq x_{n+1}. \end{aligned}$$

Thus, we see that (x_n) is decreasing. Moreover, since $x_n \geq 0$ and $x_n \leq x_1 = \frac{1}{\sqrt[3]{2}}$ we know that

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

We now need to show

$$\inf\{x_n : n \in \mathbb{N}\} = 0.$$

Given $\varepsilon > 0$, we need to show that we may find $n \in \mathbb{N}$ so that $x_n < \varepsilon$.

By the Archimedean property, we may choose

$$n > \frac{1}{\varepsilon^3} - 1.$$

Thus, in such a case

$$x_n = \frac{1}{\sqrt[3]{n+1}} < \varepsilon.$$

Hence $x_n \to 0$ as $n \to \infty$.

Homework Exercises: page 74 numbers 2, 4, 6 and 7